GATE GUIDE

Sample Chapter

SIGNALS & SYSTEMS

Theory, Example and Practice

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Features:-

- Brief and explicit theory
- Problem solving methodology
- Detailed explanations of examples
- Practice Exercises

JHUNJHUNUWALA

An Introduction to the Series GATE GUIDE

by RK Kanodia & Ashish Murolia

The GATE examination consists of multiple choice problems which are tricky, conceptual and tests the fundamental understanding of the subject. As an GATE aspirant your study should be emphasized on the following points.

Brief and explicit Theory which covers all the topics: The syllabus of GATE examination includes all the subjects of under graduation which you have to study in a short span of your preparation. The theory should be point-to-point and explicit which develops the fundamentals of the subject. Additionally, it should give you the whole coverage of the syllabus.
However, for the interview you should always refer standard text books and reference books only.

Concepts & Formulas: The question appeared in GATE are numerical as well as conceptual. The theory should include all the concepts and formulas which should be highlighted for a quick reading.

Step-by-step Problem Solving Methodology: For solving different kind of numerical problems, a particular methodology should be followed. Note that for a specific problem alternate methods can be used. The best method is one which is much simpler and less time consuming.

Well-explained examples: Solved examples gives a good understanding of the solution methodologies. They enhance the problem solving skills. Also, they make you to choose the best solution between alternate methods.

Practice Exercise: Only theory is not sufficient for a good score. You need to practice as much questions as you can. Remember that, similar questions do not give the whole breadth of the syllabus. There should be a variety of questions which covers all the topics. Questions should be numerical as well as conceptual. Questions appeared in GATE has a standard level, they can not be compared with any other exams. Questions seems simple but most of them are tricky. So, practice questions must be of the same level as GATE examination questions.

The book **GATE GUIDE** is featured with all above points. Let us have a glance of the book

AT A GLANCE

1. Brief Theory

Each chapter comprises brief theory covering all the topics. It is very explicit and provides a clear understanding of the topics.

Page 314 The Laplace Transform Chapter 5

5.1 INTRODUCTION

It is a mathematical tool that transforms any continuous-time signal into a completely different signal representation that is a function of a complex variable s. The Laplace transform can also be used to analyze LTI systems with nonzero initial conditions.

It is classified as the bilateral Laplace transform and unilateral Laplace transform. The bilateral Laplace transform is used for both the non-causal and causal signals, while the unilateral Laplace transform is defined only for causal signals.

The Laplace transform also exists for an unbounded signal or for an unstable impulse response.

5.1.1 The Bilateral or Two-Sided Laplace Transform

The bilateral or two-sided Laplace transform of a continuous-time signal $x(t)$ is defined as

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \qquad (5.1.1)$$

Where, $X(s)$ is the transformed signal and $\mathcal{L}$ represents the Laplace transformation. The complex variable s comprises a real part and an imaginary part and is expressed as

$$s = \sigma + j\omega$$

The signal $x(t)$ and its Laplace transform $X(s)$ are said to form a Laplace transform pair denoted as

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

5.1.2 The Unilateral Laplace Transform

The Laplace transform for causal signals and systems is referred to as the unilateral Laplace transform and is defined as follows

$$X(s) = \mathcal{L}\{x(t)\} = \int_{0}^{\infty} x(t)\, e^{-st}\, dt \qquad (5.1.2)$$

For causal signals and systems, the unilateral and bilateral Laplace transforms are the same.

► EXAMPLE

The Laplace transform of the signal $x(t) = e^{-at}u(t)$ is

(A) $\frac{1}{(s-a)}$ (B) $\frac{1}{(s^2-a^2)}$

Page 314, Chapter-5, The Laplace Transform

2. Solved Example (Multiple Choice)

Each topic is followed by a Multiple choice solved example which has a significant relevance with theory.

5. Marginal Notes

Marginal notes are extra source of learning. They emphasize useful concepts, summarized text and some common mistakes that students need to avoid.

3. Text Screen

The subject of Signals & System includes various properties of systems & transforms. These are highlighted in a text screen showing their importance while reading.

4. Table of Summary

The whole text is summarized using tables which helps in quick reading.

energy signal, the average power $P = 0$.

Power Signal

A DT signal $x[n]$ is defined as a power sequence, if the average power of the signal is finite i.e. $0 < P < \infty$. For a sequence with average power P being finite, the total energy $E = \infty$.

A DT signal can not be both an energy and a power signal simultaneously. Some signals, however, can be classified as neither power signals nor as energy signals.

The concept of energy signal and power signal is given in the following text screen

> Most DT periodic sequences are power signals. However all power signals need not be periodic.
> Energy signals generally include non-periodic signals that have a finite time duration. In contrast, bounded finite-duration signals are energy signals.

Summarized Table For Discrete Time Energy and Power Signals

TABLE 3.1 : Energy & Power DT Signals

	Energy Signal		Power Signal
1.	The total energy is obtained using $E = \sum_{n=-\infty}^{\infty} \lvert x[n] \rvert^2$	1.	The average power is obtained using $P = \lim_{N \to \infty} \frac{1}{(2N+1)} \sum_{n=-N}^{N} \lvert x[n] \rvert^2$
2.	For the energy signal $0 < E < \infty$, and the average power $P = 0$.	2.	For the power signal $0 < P < \infty$, and the energy $E = \infty$.
3.	Non-periodic signals are energy signals.	3.	Periodic signals are power signals. However all power signals need not be periodic.
4.	Energy signals are not time limited.	4.	Power signals exist over infinite time.

6. Problem Solving Method
a step by step approach for problem solving procedures.

$$= f_1[n] * f_2[n]$$
$$= x[n] * y[-n]$$

4.6.5 Methods to solve correlation

Graphical Method

The correlation of two signals $x[n]$ and $y[n]$ can be calculated using graphical method, explained in the following methodology

METHODOLOGY

1. Obtain the sequence $y[n-k]$ by shifting the sequence $y[n]$ to the right by a time lag k.
2. Multiply the shifted sequence $y[n-k]$ with $x[n]$ and sum all the values to obtain $R_{xy}[k]$.
3. Repeat steps 1 and 2 for all values of the lag k.

Matrix Method

From section 4.6.4, we know that convolution of $x[n]$ and folded sequence $y[-n]$ gives correlation sequence $R_{xy}[n]$. Hence the correlation can be obtained by the matrix method of convolution which is same as given in section 4.2.1. This methodology is given below

METHODOLOGY

1. Write down the sequences $x[n]$ and folded sequence $y[-n]$ as the column and row of the matrix. Mark ↑ at the origins of $x[n]$ and $y[-n]$.
2. Multiply each and every sample in $x[n]$ with the samples of $y[-n]$ and complete the matrix.
3. Divide the elements in the matrix by drawing diagonal lines as shown in the example.
4. Starting from left sum all the elements in each strip and write down in the same order. Sum of elements in each strip n corresponds to the correlation sequence $R_{xy}[n]$.

7. Practice Exercise

Practice exercises covers variety of problems from each topic that enhance your confidence level. Practice exercises are divided into two levels on the basis of complexity.

PRACTICE EXERCISE

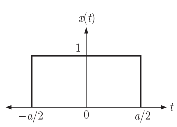
LEVEL-1

MCQ 7.1 The Fourier transform of a CT signal $x(t) = \exp\left(-a\left|t\right|\right)$, $a > 0$ is

(A) $\frac{-a}{a^2+\omega^2}$ (B) $\frac{2a}{a^2+\omega^2}$

(C) $\frac{1}{a^2+\omega^2}$ (D) $\frac{-1}{a^2+\omega^2}$

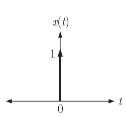
MCQ 7.2 The Fourier transform of the function $x(t)$ shown in figure is



(A) $\frac{2}{\omega}\sin(\omega a)$ (B) $\frac{2}{\omega a}\sin\left(\frac{\omega a}{2}\right)$

(C) $\frac{2}{\omega}\sin\left(\frac{\omega a}{2}\right)$ (D) $\frac{2}{\omega a}\sin\left(\frac{\omega a}{4}\right)$

MCQ 7.3 Match List I (CT signal) with List II (Fourier transform) and choose the correct answer using the codes given below

List I (CT Signal)	**List II** (Fourier Transform)
P. 	1. $\frac{\sin(0.5\omega)}{0.5\omega}$

PRACTICE EXERCISE

LEVEL-2

MCQ 7.1 The Fourier transform of signal $e^{-2t}u(t-3)$ is

(A) $\frac{e^{-3(2-j\omega)}}{2-j\omega}$ (B) $\frac{e^{-3(2+j\omega)}}{2+j\omega}$

(C) $\frac{e^{3(2-j\omega)}}{2-j\omega}$ (D) $\frac{e^{3(2+j\omega)}}{2+j\omega}$

MCQ 7.2 The Fourier transform of signal $\delta(t+1)+\delta(t-1)$ is

(A) $\frac{2}{1+j\omega}$ (B) $\frac{2}{1-j\omega}$

(C) $2\cos\omega$ (D) None of the above

MCQ 7.3 The Fourier transform of signal $te^{-1}u(t)$ is

(A) $\frac{1}{1+\omega^2}$ (B) $\frac{-1}{1+\omega^2}$

(C) $\frac{1}{(1+j\omega)^2}$ (D) $\frac{1}{(1-j\omega)^2}$

MCQ 7.4 The Fourier transform of signal $\sum_{m=0}^{\infty} a^m\delta(t-m)$, $\left|a\right| < 1$, is

(A) $\frac{a}{1-ae^{-j\omega}}$ (B) $\frac{a}{1+ae^{-j\omega}}$

(C) $\frac{1}{1+ae^{-j\omega}}$ (D) $\frac{1}{1-ae^{-j\omega}}$

MCQ 7.5 The Fourier transform of signal $sgn(t)$ is

(A) $\frac{-2}{j\omega}$ (B) $\frac{4}{j\omega}$

(C) $\frac{2}{j\omega}$ (D) $\frac{1}{j\omega}+1$

MCQ 7.6 The Fourier transform of signal $u(t)$ is

(A) $\pi\delta(\omega)$ (B) $1/j\omega$

(C) $\pi\delta(\omega)+(1/j\omega)$ (D) None of the above

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SYLLABUS

GATE ELECTRONICS & COMMUNICATION ENGINEERING

SIGNALS & SYSTEMS

Definitions and properties of Laplace transform, continuous-time and discrete-time Fourier series, continuous-time and discrete-time Fourier Transform, DFT and FFT, Z-transform. Sampling theorem. Linear Time-Invariant (LTI) Systems: definitions and properties; causality, stability, impulse response, convolution, poles and zeros, parallel and cascade structure, frequency response, group delay, phase delay. Signal transmission through LTI systems.

GATE ELECTRICAL ENGINEERING

SIGNALS & SYSTEMS

Representation of continuous and discrete-time signals; shifting and scaling operations; linear, time-invariant and causal systems; Fourier series representation of continuous periodic signals; sampling theorem; Fourier, Laplace and Z- transform.

GATE INSTRUMENTATION ENGINEERING

SIGNALS, SYSTEMS & COMMUNICATION*

Periodic and aperiodic signals. Impulse response, transfer function and frequency response of first- and second order systems. Convolution, correlation and characteristics of linear time invariant systems. Discrete time system, impulse and frequency response. Pulse transfer function. IIR and FIR filters. Amplitude and frequency modulation and demodulation. Sampling theorem, pulse code modulation. Frequency and time division multiplexing. Amplitude shift keying, frequency shift keying and pulse shift keying for digital modulation.

*communication part is not covered in the book.

IES ELECTRONICS & TELECOMMUNICATION ENGINEERING

SIGNALS & SYSTEMS

Classification of signals and systems: System modelling in terms of differential and difference equations; State variable representation; Fourier series; Fourier transforms and their application to system analysis; Laplace transforms and their application to system analysis; Convolution and superposition integrals and their applications; Z-transforms and their applications to the analysis and characterisation of discrete time systems; Random signals and probability, Correlation functions; Spectral density; Response of linear system to random inputs.

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THE Z-TRANSFORM

CHAPTER OUTLINE

6.1 INTRODUCTION

As we studied in previous chapter, the Laplace transform is an important tool for analysis of continuous time signals and systems. Similarly, z-transforms enables us to analyze discrete time signals and systems in the z-domain.

The properties of z-transform are similar to those of the Laplace transform.

Like, the Laplace transform, it is also classified as bilateral z-transform and unilateral z-transform.

The bilateral or two-sided z-transform is used to analyze both causal and non-causal LTI discrete systems, while the unilateral z-transform is defined only for causal signals.

6.1.1 The Bilateral or Two-Sided z-transform

The z-transform of a discrete-time sequence $x[n]$, is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad (6.1.1)$$

Where, $X(z)$ is the transformed signal and $\mathcal{Z}$ represents the z-transformation. z is a complex variable. In polar form, z can be expressed as

$$z = re^{j\Omega}$$

The signal $x[n]$ and its z-transform $X(z)$ are said to form a z-transform pair denoted as

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

where r is the magnitude of z and Ω is the angle of z. This corresponds to a circle in z plane with radius r as shown in figure 6.1.1 below

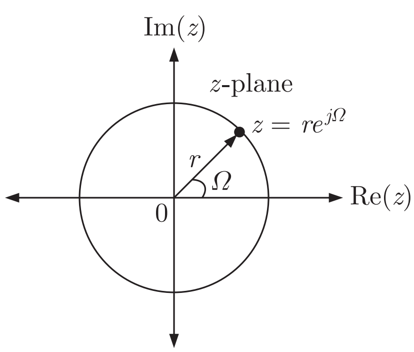


Fig 6.1.1 z-plane

6.1.2 The Unilateral or One-sided z-transform

The z-transform for causal signals and systems is referred to as the unilateral z-transform. For a causal sequence

$$x[n] = 0, \text{ for } n < 0$$

Therefore, the unilateral z-transform is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} \qquad (6.1.2)$$

For causal signals and systems, the unilateral and bilateral z-transform are the same.

► E X A M P L E

The bilateral z-transform of sequence $x[n] = -a^n u[-n-1]$ will be

(A) $\dfrac{1}{(1-az^{-1})}$ (B) $\dfrac{a}{(z-a)}$

(C) $\dfrac{-1}{(1-az^{-1})}$ (D) $\dfrac{1}{(z-a)}$

SOLUTION :

The bilateral z-transform of $x[n]$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = -\sum_{n=-\infty}^{\infty} a^n u[-n-1]\, z^{-n}$$

We know that

$$u[-n-1] = \begin{cases} 1, & \text{for } -n-1 \geq 0 \text{ or } n \leq -1 \\ 0, & n > -1 \end{cases}$$

So $\quad X(z) = -\sum_{n=-\infty}^{-1} (az^{-1})^n$

substituting $n = -k$

$$= -\sum_{k=1}^{\infty} (az^{-1})^{-k} = -\sum_{k=1}^{\infty} (a^{-1}z)^k$$

$$= \frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}}$$

Hence (A) is correct option.

► E X A M P L E

The unilateral z-transform of sequence $x[n] = \{\underset{\uparrow}{1}, 2, 2, 1\}$ is equal to

(A) $1 + 2z + 2z^2 + z^3$ (B) $1 + \dfrac{2}{z} + \dfrac{2}{z^2} + \dfrac{1}{z^3}$

(C) $z^3 + 2z^2 + 2z^{-1} + \dfrac{1}{z}$ (D) $\dfrac{1}{z} + \dfrac{2}{z^2} + \dfrac{2}{z^3} + \dfrac{1}{z^4} + 1$

SOLUTION :

The unilateral z-transform of sequence $x[n]$ is given by

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$
$$= \sum_{n=0}^{3} x[n] z^{-n}$$
$$= x[0] z^{0} + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3}$$
$$= 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$
$$= 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{1}{z^3}$$

Hence (B) is correct option.

6.2 Existence of z-Transform

Consider the bilateral z-transform given by equation (6.1.1)

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

The z-transform exists when the infinite sum in above equation converges. For this summation to be converged $|x[n] z^{-n}|$ must be absolutely summable.

Substituting $z = re^{j\Omega}$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] (re^{j\Omega})^{-n}$$

or,

$$X[z] = \sum_{n=-\infty}^{\infty} \{x[n] r^{-n}\} e^{-j\Omega n}$$

Thus for existence of z-transform

$$|X(z)| < \infty$$

$$\sum_{n=-\infty}^{\infty} x[n] r^{-n} < \infty \qquad (6.2.1)$$

6.3 Region of Convergence

The existence of z-transform is given from equation (6.2.1). The values of r for which $x[n] r^{-n}$ is absolutely summable is referred to as region of convergence. Since, $z = re^{j\Omega}$ so $r = |z|$. Therefore we conclude that the range of values of the variable $|z|$ for which the sum in equation (6.1.1) converges is called the region of convergence. This can be explained through the following examples.

► EXAMPLE

The Region of convergence for the z-transform of sequence $x[n] = -a^n u[-n-1]$ will be

(A) $|z| > |a|$ (B) $|z| > 0$

(C) $|z| < |a|$ (D) $|z| < 0$

SOLUTION :

As solved in example (1), z-transform of $x[n]$ is

$$X(z) = -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{k=1}^{\infty} (az^{-1})^{-k}$$

$$= -\sum_{k=1}^{\infty} (a^{-1}z)^k$$

$$= -[a^{-1}z + (a^{-1}z)^2 + (a^{-3}z)^3 + ...]$$

This series converges if $|a^{-1}z| < 1$ or $|z| < |a|$

Hence (C) is correct option.

► EXAMPLE

The region of convergence of z-transform of sequence $x[n] = a^n u[n]$ is

(A) $|z| < a$ (B) $|z| > a$

(C) $|z| > 0$ (D) entire z-plane

SOLUTION :

The z-transform of sequence $a^n u[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$\because$ $$u[n] = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

so,

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= 1 + (az^{-1}) + (az^{-1})^2 + ...$$

$$= \frac{1}{1 - az^{-1}}$$

This series converges if $|az^{-1}| < 1$

or $|z| > |a|$

Thus ROC of $X(z)$ is $|z| > |a|$

Hence (B) is correct option.

Note : In example (3) and (4) we have seen that z-transform of $-a^n u[-n-1]$ and $a^n u[n]$ is same but ROC of transform is different for both. Thus, z-transform of a sequence is completely specified if both the expression $[X(z)]$ and ROC are given to us.

6.3.1 Poles & Zeros of Rational z-transforms

The most common form of z-transform is a rational function. Let $X(z)$ be the z-transform of sequence $x[n]$, expressed as a ratio of two polynomials $N(z)$ and $D(z)$.

$$X(z) = \frac{N(z)}{D(z)}$$

The roots of numerator polynomial i.e. values of z for which $X(z) = 0$ is referred to as zeros of $X(z)$. The roots of denominator polynomial for which $X(z) = \infty$ is referred to as poles of $X(z)$. The representation of $X(z)$ through its poles and zeros in the z-plane is called pole-zero plot of $X(z)$.

In pole-zero plot poles are marked by a small cross '×' and zeros are marked by a small dot 'o' as shown in figure 6.3.1.

For example consider a rational transfer function $X(z)$ given as

$$H(z) = \frac{z}{z^2 - 5z + 6}$$

$$= \frac{z}{(z-2)(z-3)}$$

Now, the zeros of $X(z)$ are roots of numerator that is $z = 0$ and poles are roots of equation $(z-2)(z-3) = 0$ which are given as $z = 2$ and $z = 3$. The poles and zeros of $X(z)$ are shown in pole-zero plot of figure 6.3.1.

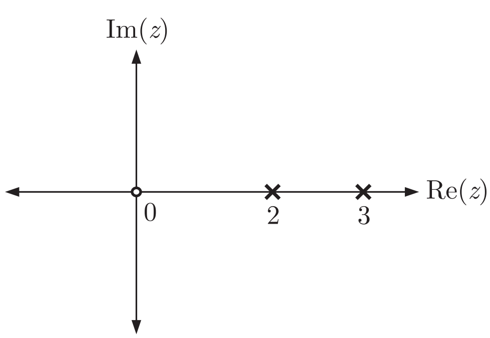


Fig 6.3.1 Pole-zero plot of $X(z)$

6.3.2 Properties of ROC

The various properties of ROC are summarized as follows. These properties can be proved by taking appropriate examples of different DT signals.

Property 1 : The ROC is a concentric ring in the z -plane centered about the origin.

Proof :

The z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Put $z = re^{j\Omega}$

$$X(z) = X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, r^{-n}\, e^{-j\Omega n}$$

$X(z)$ converges for those values of z for which $x[n]\, r^{-n}$ is absolutely summbable that is

$$\sum_{n=-\infty}^{\infty} x[n]\, r^{-n} < \infty$$

Thus, convergence is dependent only on r, where, $r = |z|$
The equation $z = re^{j\Omega}$, describes a circle in z-plane. Hence the ROC will consists of concentric rings centered at zero.

Property 2 : The ROC cannot contain any poles.

Proof :

ROC is defined as the values of z for which z-transform $X(z)$ converges. We know that $X(z)$ will be infinite at pole, and, therefore $X(z)$ does not converge at poles. Hence the region of convergence does not include any pole.

Property 3 : If $x[n]$ is a finite duration two-sided sequence then the ROC is entire z-plane except at $z = 0$ and $z = \infty$.

Proof :

A sequence which is zero outside a finite interval of time is called 'finite duration sequence'. Consider a finite duration sequence $x[n]$ shown in figure 6.3.2a; $x[n]$ is non-zero only for some interval $N_1 \le n \le N_2$.

Both N_1 and N_2 can be either positive or negative.

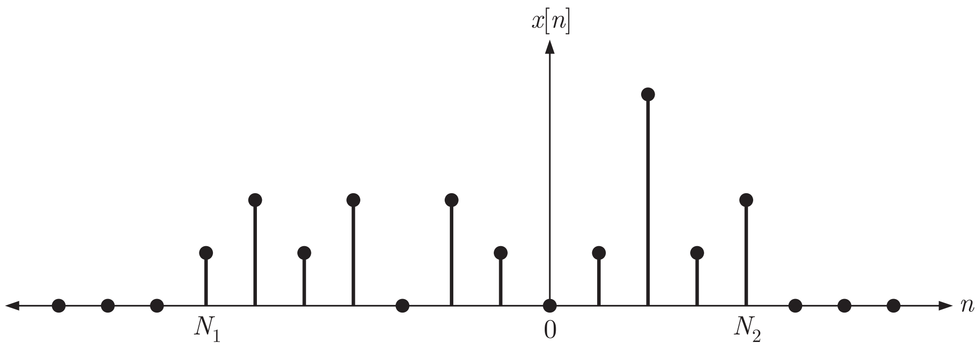


Fig 6.3.2a A finite duration sequence

The z-transform of $x[n]$ is defined as

$$X(z) = \sum_{n=N_1}^{N_2} x[n]\, z^{-n}$$

This summation converges for all finite values of z. If N_1 is negative and N_2 is positive, then $X(z)$ will have both positive and negative powers of z. The negative powers of z becomes unbounded (infinity) if $|z| \to 0$. Similarly positive powers of z becomes unbounded (infinity) if $|z| \to \infty$. So ROC of $X(z)$ is entire z-plane except possible $z = 0$ and/or $z = \infty$.

Property 4 : If $x[n]$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $|z| > r_0$ will also be in the ROC.

Proof :

A sequence which is zero prior to some finite time is called the 'right-sided sequence'. Consider a right-sided sequence $x[n]$ shown in figure 6.3.2b; that is;

$$x[n] = 0 \text{ for } n < N_1.$$

Here N_1 can be either positive or negative.

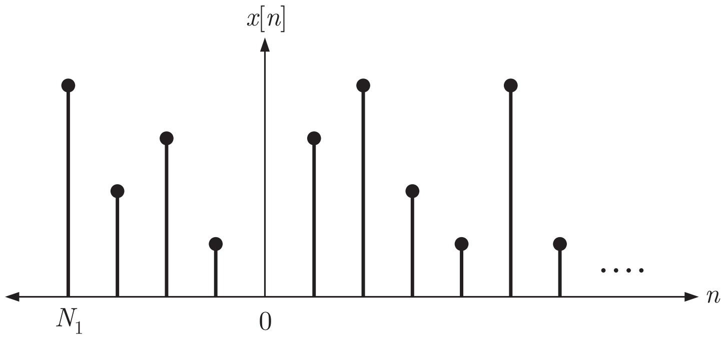


Fig 6.3.2b A right-sided sequence

Let the z-transform of $x[n]$ converges for some value of $|z|\left(\text{i.e. } |z| = r_0\right)$. From the condition of convergence we can write

$$\left|\sum_{n=-\infty}^{\infty} x[n]\, z^{-n}\right| < \infty$$

$$\sum_{n=-\infty}^{\infty} |x[n]| r_0^{-n} < \infty$$

The sequence is right sided, so limits of above summation changes as

$$\sum_{n=N_1}^{\infty} |x[n]| r_0^{-n} < \infty \qquad (6.3.1)$$

now if we take another value of z as $|z| = r_1$ with $r_1 < r_0$, then $x[n]\, r_1^{-n}$ decays faster than $x[n]\, r_0^{-n}$ for increasing n. Thus we can write

$$\sum_{n=N_1}^{\infty} |x[n]| z^{-n} = \sum_{n=N_1}^{\infty} |x[n]| z^{-n} r_0^{-n} r_0^{n}$$

$$= \sum_{n=N_1}^{\infty} |x[n]| r_0^{-n} \left(\frac{z}{r_0}\right)^{-n} \qquad (6.3.2)$$

From equation (6.3.1) we know that $x[n]\, r_0^{-n}$ is absolutely summable. Let, it is bounded by some value M_x, then equation (6.3.2) becomes as

$$\sum_{n=N_1}^{\infty} |x[n]| z^{-n} \le M_x \sum_{n=N_1}^{\infty} \left(\frac{z}{r_0}\right)^{-n} \qquad (6.3.3)$$

The right hand side of above equation converges only if

$$\left|\frac{z}{r_0}\right| > 1 \text{ or } |z| > r_0$$

Thus, we conclude that if the circle $|z| = r_0$ is in the ROC, then all values of z for which $|z| > r_0$ will also be in the ROC. The ROC of a right-sided sequence is illustrated in figure 6.3.2c.

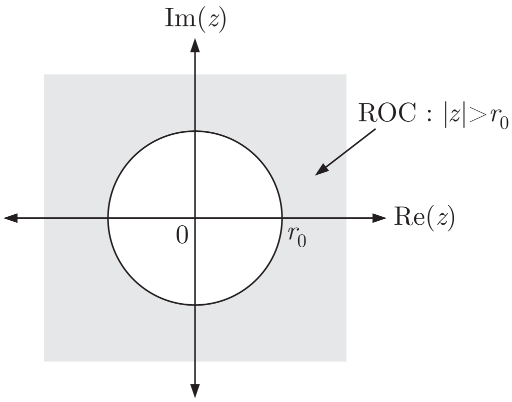


Fig 6.3.2c ROC of a right-sided sequence

Property 5 : If $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $|z| < r_0$ will also be in the ROC.

Proof :

A sequence which is zero after some finite time interval is called a 'left-sided signal'. Consider a left-sided signal $x[n]$ shown in figure 6.3.2d; that is $x[n] = 0$ for $n > N_2$.

Here N_2 can be either positive or negative.

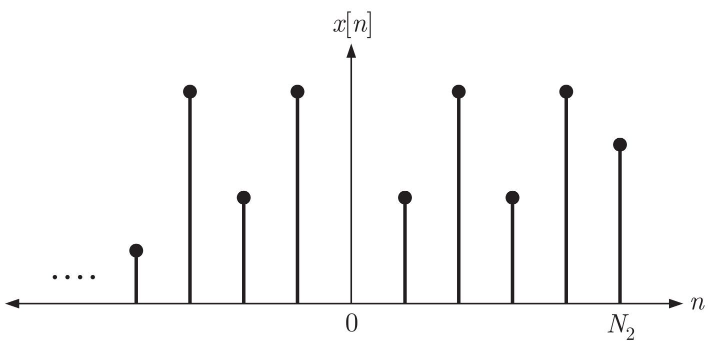


Fig 6.3.2d A left-sided sequence

Let z-transform of $x[n]$ converges for some values of $|z|$ $\left(\text{i.e.} |z| = r_0\right)$. From the condition of convergence we write

$$\left|\sum_{n=-\infty}^{\infty} x[n]\, z^{-n}\right| < \infty$$

or
$$\sum_{n=-\infty}^{\infty} |x[n]|\, r_0^{-n} < \infty \tag{6.3.4}$$

The sequence is left sided, so the limits of summation changes as

$$\sum_{n=-\infty}^{N_2} |x[n]| r_0^{-n} < \infty \qquad (6.3.5)$$

now if take another value of z as $|z| = r_1$, then we can write

$$\sum_{n=-\infty}^{N_2} |x[n]| z^{-n} = \sum_{n=-\infty}^{N_2} |x[n]| z^{-n} r_0^{-n} r_0^{n}$$

$$= \sum_{n=-\infty}^{N_2} |x[n]| r_0^{-n} \left(\frac{r_0}{z}\right)^n \qquad (6.3.6)$$

From equation (6.3.4), we know that $x[n]\, r_0^{-n}$ is absolutely summable. Let it is bounded by some value M_x, then equation (6.3.6) becomes as

$$\sum_{n=-\infty}^{N_2} |x[n]| z^{-n} \leq M_x \sum_{n=-\infty}^{N_2} \left(\frac{r_0}{z}\right)^n$$

the above summation converges if $\left|\frac{r_0}{z}\right| > 1$ (because n is increasing negatively), so $|z| < r_0$ will be in ROC.

The ROC of a left-sided sequence is illustrated in figure 6.3.2e.

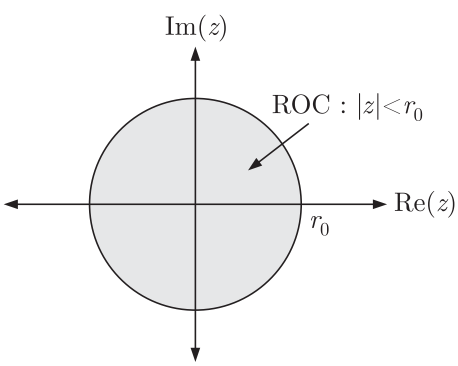


Fig 6.3.2e ROC of a left-sided sequence

Property 6 : If $x[n]$ is a two-sided signal, and if the circle $|z| = r_0$ is in the ROC, then the ROC consists of a ring in the z-plane that includes the circle $|z| = r_0$

Proof :

A sequence which is defined for infinite extent for both $n > 0$ and $n < 0$ is called 'two-sided sequence'. A two-sided signal $x[n]$ is shown in figure 6.3.2f.

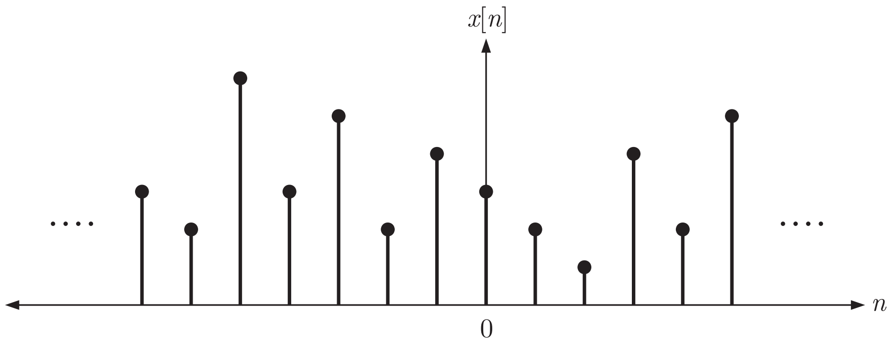


Fig 6.3.2f A two-sided sequence

For any time N_0, a two-sided sequence can be divided into sum of left-sided and right-sided sequences as shown in figure 6.3.2g.

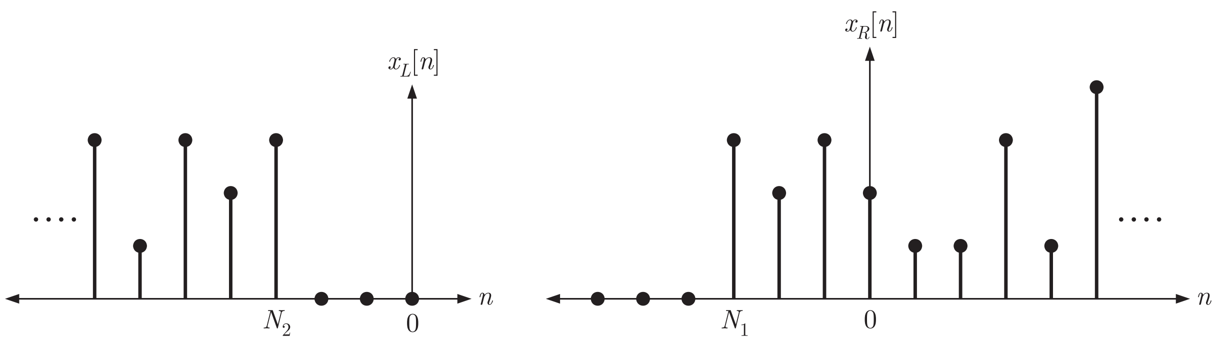


Fig 6.3.2g A two sided sequence divided into sum of a left-sided and right-sided sequence

The z-transform of $x[n]$ converges for the values of z for which the transform of both $x_R[n]$ and $x_L[n]$ converges. From property 4, the ROC of a right-sided sequence is a region which is bounded on the inside by a circle and extending outward to infinity i.e. $|z| > r_1$. From property 5, the ROC of a left sided sequence is bounded on the outside by a circle and extending inward to zero i.e. $|z| < r_2$. So the ROC of combined signal includes intersection of both ROCs which is ring in the z-plane.

The ROC for the right-sided sequence $x_R[n]$, the left-sequence $x_L[n]$ and their combination which is a two sided sequence $x[n]$ are shown in figure 6.3.2h.

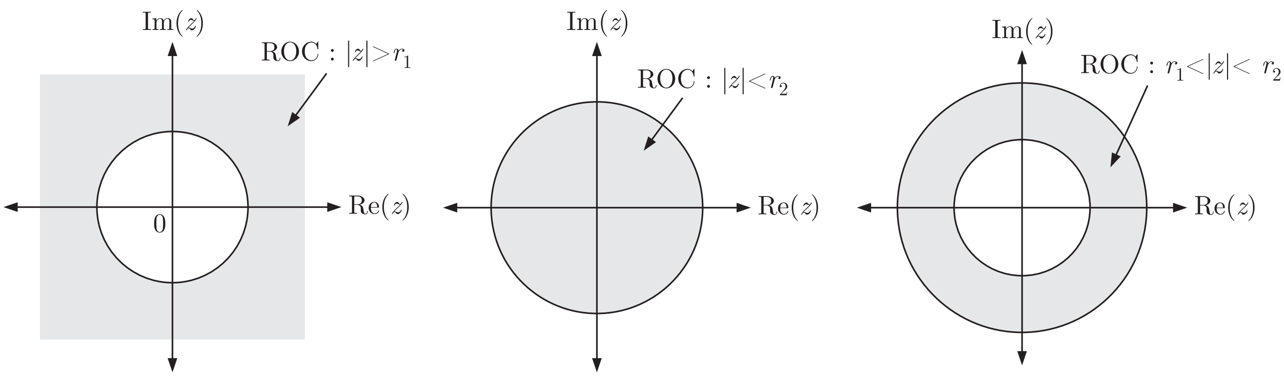


Fig 6.3.2h ROC of a left-sided sequence, a right-sided sequence and two sided sequence

Property 7 : If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.

Proof : The exponential DT signals also have rational z -transform and the poles of $X(z)$ determines the boundaries of ROC.

Property 8 : If the z-transform $X(z)$ of $x[n]$ is rational and $x[n]$ is a right-sided sequence then the ROC is the region in the z-plane outside the outermost pole i.e. ROC is the region outside a circle with a radius greater than the magnitude of largest pole of $X(z)$.

Proof :

This property can be be proved by taking property 4 and 7 together.

▶ EXAMPLE

The region of convergence of the z-transform of sequence

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \text{ is}$$

(A) $|z| < \frac{1}{2}$

(B) $\frac{1}{3} < |z| < \frac{1}{2}$

(C) $|z| < \frac{1}{3}$

(D) $|z| > \frac{1}{2}$

SOLUTION :

The z-transform of sequence $x[n]$ is obtained as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]\, z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n]\, z^{-n}$$

$$= \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n}_{\text{I}} + \underbrace{\sum_{n=0}^{\infty} \left(-\frac{1}{3z}\right)^n}_{\text{II}} = \frac{2\left(2z - \frac{1}{6}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}$$

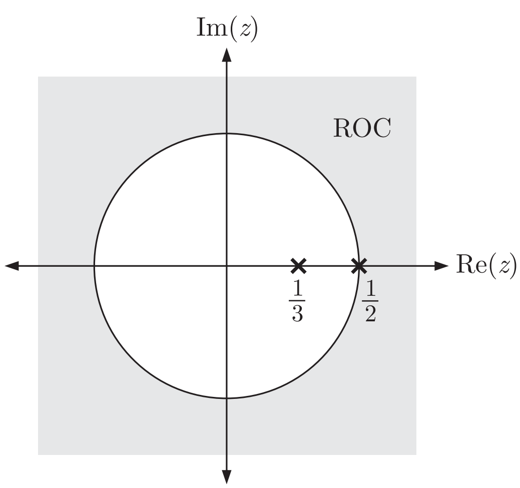
Poles are $z = 1/2,\ z = -1/3$

summation I converges if $\left|\frac{1}{2z}\right| < 1$ or $|z| > \frac{1}{2}$

summation II converges if $\left|\frac{1}{3z}\right| < 1$ or $|z| > \frac{1}{3}$

ROC is intersection of above two conditions so

ROC : $|z| > \frac{1}{2}$ (which is outside the outermost pole)



Hence (D) is correct Option.

Property 9 : If the z-transform $X(z)$ of $x[n]$ is rational and $x[n]$ is a left-sided sequence then the ROC is the region in the z-plane inside the innermost pole i.e. ROC is the region inside a circle with a radius equal to the smallest magnitude of poles of $X(z)$.

Proof :

This property can be be proved by taking property 5 and 7 together.

▶ EXAMPLE

The region of convergence of the z-transform of sequence

$$x[n] = \left(-\frac{1}{2}\right)^n u[-n-1] - \left(-\frac{1}{3}\right)^n u[-n-1] \text{ is}$$

(A) $|z| < \frac{1}{3}$

(B) $\frac{1}{2} < |z| < \frac{1}{3}$

(C) $|z| > \frac{1}{2}$

(D) $|z| < \frac{1}{2}$

SOLUTION :

z-transform of $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= -\sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[-n-1]\, z^{-n}$$

$$- \sum_{n=\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[-n-1]\, z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n} - \sum_{n=-\infty}^{-1} \left(-\frac{1}{3}\right)^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{-n} z^n - \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{-n} z^n$$

$$= \underbrace{-\sum_{n=1}^{\infty} (2z)^n}_{\text{I}} - \underbrace{\sum_{n=1}^{\infty} (-3z)^n}_{\text{II}}$$

$$= -\frac{2z}{(1-2z)} - \frac{(-3z)}{(1+3z)}$$

$$= \frac{-2z(1+3z) + 3z(1-2z)}{(1-2z)(1+3z)}$$

$$\frac{(z-12z^2)}{(1-2z)(1+3z)}$$

$$= \frac{z(1-12z)}{-2\left(z-\frac{1}{2}\right)(3)\left(z+\frac{1}{3}\right)}$$

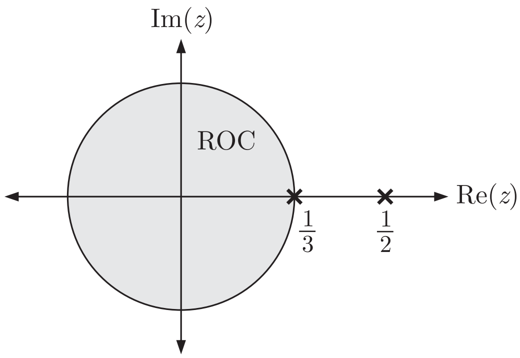
$$= \frac{z\left(2z - \frac{1}{6}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}$$

Poles are $z = \frac{1}{2},\ z = -\frac{1}{3}$

ROC : Summation I converges if $|2z| < 1$ or $|z| < \frac{1}{2}$

summation II converges it $|3z| < 1$ or $|z| < \frac{1}{3}$

ROC is intersection of both so $|z| < \frac{1}{3}$
(which is inside the innermost pole)



Hence (A) is correct Option.

z-Transform of Some Basic Functions

Z-transform of basic functions are summarized in the following table with their respective ROCs.

TABLE 6.1 : z-Transform of Basic Discrete Time Signals

	DT sequence $x[n]$	**z-transform**	**ROC**
1.	$\delta[n]$	1	entire z -plane
2.	$\delta[n - n_0]$	z^{-n_0}	entire z -plane, except $z = 0$

3.	$u[n]$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\|z\| > 1$
4.	$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$	$\|z\| > \|\alpha\|$
5.	$\alpha^{n-1} u[n-1]$	$\frac{z^{-1}}{1-\alpha z^{-1}} = \frac{1}{z-\alpha}$	$\|z\| > \|\alpha\|$
6.	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$	$\|z\| > 1$
7.	$n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} = \frac{\alpha z}{(z-\alpha)^2}$	$\|z\| > \alpha$
8.	$\cos(\Omega_0 n)\, u[n]$	$\frac{1 - z^{-1}\sin\Omega_0}{1 - 2z^{-1}\cos\Omega_0 + z^{-2}}$ or $\frac{z[z-\cos\Omega_0]}{z^2 - 2z\cos\Omega_0 + 1}$	$\|z\| > 1$
9.	$\sin(\Omega_0 n)\, u[n]$	$\frac{z^{-1}\sin\Omega_0}{1 - 2z^{-1}\cos\Omega_0 + z^{-2}}$ or $\frac{z\sin\Omega_0}{z^2 - 2z\cos\Omega_0 + 1}$	$\|z\| > 1$
10.	$\alpha^n \cos(\Omega_0 n)\, u[n]$	$\frac{1 - \alpha z^{-1}\cos\Omega_0}{1 - 2\alpha z^{-1}\cos\Omega_0 + \alpha^2 z^{-2}}$ or $\frac{z[z-\alpha\cos\Omega_0]}{z^2 - 2\alpha z\cos\Omega_0 + \alpha^2}$	$\|z\| > \|\alpha\|$
11.	$\alpha^n \sin(\Omega_0 n)\, u[n]$	$\frac{\alpha z^{-1}\sin\Omega_0}{1 - 2\alpha z^{-1}\cos\Omega_0 + \alpha^2 z^{-2}}$ or $\frac{\alpha z\sin\Omega_0}{z^2 - 2\alpha z\cos\Omega_0 + \alpha^2}$	$\|z\| > \alpha$
12.	$r\alpha^n \sin(\Omega_0 n + \theta)\, u[n]$ with $\alpha \in R$	$\frac{A + Bz^{-1}}{1 + 2\gamma z^{-1} + \alpha^2 z^{-2}}$ or $\frac{z(Az+B)}{z^2 + 2\gamma z + \gamma^2}$	$\|z\| \leq \|\alpha\|$[(a)]

6.4 THE INVERSE z-TRANSFORM

Let $X(z)$ be the z-transform of a sequence $x[n]$. To obtain the sequence $x[n]$ from its z-transform is called the inverse z-transform. The inverse z-transform is given as

$$x[n] = \frac{1}{2\pi j}\oint X(z)\,z^{n-1}dz$$

This method involves the contour integration, so difficult to solve. There are other commonly used methods to evaluate the inverse z-transform given as follows

1. Partial fraction method
2. Power series expansion

6.4.1 Partial fraction method

If $X(z)$ is a rational function of z then it can be expressed as follows.

$$X(z) = \frac{N(z)}{D(z)}$$

It is convenient if we consider $X(z)/z$ rather than $X(z)$ to obtain the inverse z-transform by partial fraction method.

Let $p_1,\ p_2,\ p_3....p_n$ are the roots of denominator polynomial, also the poles of $X(z)$. Then, using partial fraction method $X(z)/z$ can be expressed as

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\frac{A_3}{z-p_3}+...+\frac{A_n}{z-p_n}$$

$$X(z) = A_1\frac{z}{z-p_1}+A_2\frac{z}{z-p_2}+...+\frac{z}{z-p_n}$$

Now, the inverse z-transform of above equation can be obtained by comparing each term with the standard z -tranform pair given in table 6.1. The values of coefficients $A_1,\ A_2,\ A_3....A_n$ depends on whether the poles are real & distinct or repeated or complex. Three cases are given as follows

Case I : Poles are simple and real

$X(z)/z$ can be expanded in partial fraction as

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\frac{A_3}{z-p_3}+...+\frac{A_n}{z-p_n} \qquad (6.4.1)$$

where A_1, A_2,... A_n are calculated as follows

$$A_1 = (z - p_1)\frac{X(z)}{z}\Big|_{z = p_1}$$

$$A_2 = (z - p_2)\frac{X(z)}{z}\Big|_{z = p_2}$$

In general,

$$A_i = (z - p_i)X(z)\big|_{z = p_i} \tag{6.4.2}$$

Case II : If poles are repeated

In this case $X(z)/z$ has a different form. Let p_k be the root which repeats l times, then the expansion of equation must include terms

$$\frac{X(z)}{z} = \frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + ...$$
$$+ \frac{A_{ik}}{(z - p_k)^i} + ... + \frac{A_{lk}}{(z - p_k)^l} \tag{6.4.3}$$

The coefficient A_{ik} are evaluated by multiplying both sides of equation (6.4.3) by $(z - p_k)^l$, differentiating $(l - i)$ times and then evaluating the resultant equation at $z = p_k$.
Thus,

$$C_{ik} = \frac{1}{\underline{|(l - i)}}\frac{d^{l-i}}{dz^{l-i}}\left[(z - p_k)^l\frac{X(z)}{z}\right]\Big|_{z = p_k} \tag{6.4.4}$$

Case III : Complex poles

If $X(z)$ has complex poles then partial fraction of the $X(z)/z$ can be expressed as

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_1^*}{z - p_1^*} \tag{6.4.5}$$

where A_1^* is complex conjugate of A_1 and p_1^* is complex conjugate of z_1. The coefficients are obtained by equation (6.4.2)

► EXAMPLE

Let $X(z)$ be the z-transform of a sequence $x[n]$ given as following

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Match List I (ROC of $X(z)$) with List II (corresponding sequence $x[n]$) and select the correct answer using the codes given below

	List I (ROC)		**List II ($x[n]$)**
P.	$\lvert z \rvert > 1$	**1.**	$[2-(0.5)^n]\,u[-n]$
Q.	$\lvert z \rvert < 0.5$	**2.**	$-2u[-n-1]-(0.5)^n u[n]$
R.	$0.5 < \lvert z \rvert < 1$	**3.**	$[-2+(0.5)^n]\,u[-n-1]$
		4.	$[2-(0.5)^n]\,u[n]$

Codes :

	P	Q	R
(A)	4	3	2
(B)	2	3	4
(C)	1	2	4
(D)	4	3	1

SOLUTION :

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$X(z) = \frac{z^2}{z^2-1.5z+0.5}$$

To use partial fraction method, consider $X(z)/z$

$$\frac{X(z)}{z} = \frac{z}{z^2-1.5z+0.5} = \frac{z}{(z-1)(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

Since poles are simple and real. So $\frac{X(z)}{z}$ can be expanded in partial fraction as

$$\frac{X(z)}{z} = \frac{A_1}{z-1}+\frac{A_2}{z-0.5}$$

$$A_1 = (z-1)\frac{X(z)}{z}\Big|_{z=1}$$

$$= (z-1)\frac{1}{(z-1)(1-0.5)} = 2$$

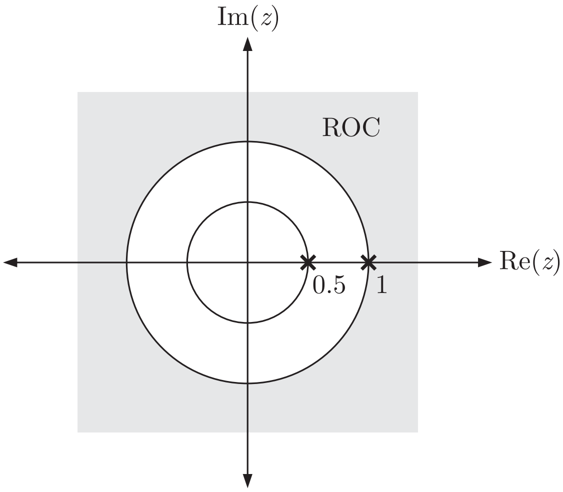
$$A_2 = (z-0.5)\frac{X(z)}{z}\Big|_{z=0.5}$$

$$= (z-0.5)\frac{0.5}{(0.5-1)(z-0.5)} = -1$$

So,
$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$
$$X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$
$$= \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

ROC : $|z| > 1$

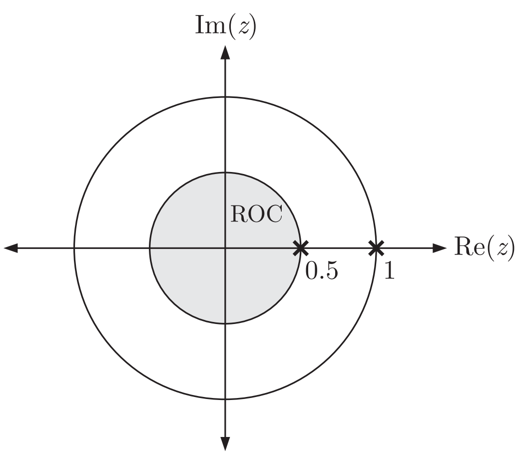
Since ROC is right to the right most pole so both the terms in equation (1) corresponds to right-sided sequence. (Refer property # 8, section 6.3)



$$\frac{2}{1-z^{-1}} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} 2(1)^n u[n]$$
$$\frac{1}{1-0.5z^{-1}} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} (0.5)^n u[n]$$

So
$$x[n] = [2-(0.5)^n]\, u[n]$$

ROC : $|z| < 0.5$



Since ROC is left to the leftmost pole so both the terms in equation (1) corresponds to a left-sided sequences. (Property # 9, section 6.3)

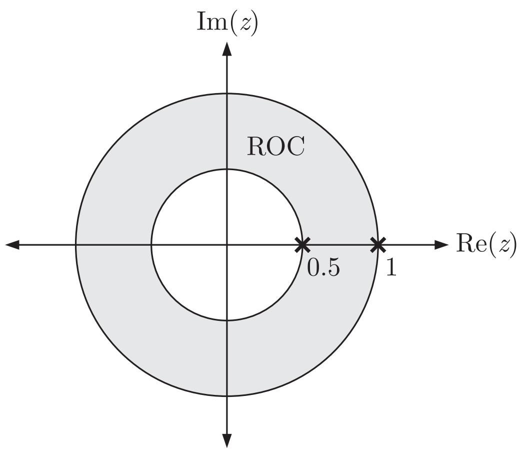
$$\frac{2}{1-z^{-1}} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} -2u[-n-1]$$

$$\frac{1}{1-0.5z^{-1}} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} -(0.5)^n u[-n-1]$$

So

$$\begin{aligned} x[n] &= -2u[-n-1] - [(-0.5)^n u[-n-1]] \\ &= -2u[-n-1] + (0.5)^n u[-n-1] \\ &= [-2 + (0.5)^n] u[-n-1] \end{aligned}$$

ROC : $0.5 < |z| < 1$



Since ROC has a greater radius than the pole at $z = 0.5$. So the second term in equation (i) corresponds the right-sided sequence, that is

$$\frac{1}{1-0.5z^{-1}} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} (0.5)^n u[n]$$

ROC $|z| < 1$, which is left to the pole at $z = 1$. So this terms will corresponds to a left sided equation.

$$\frac{2}{1-z^{-1}} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} -2u[-n-1]$$

So $$x[n] = -2u[-n-1] - (0.5)^n u[n]$$

Hence (A) is correct option.

6.4.2 Power series expansion Method

Power series method is also convenient in calculating the inverse z-transform. The z-transform of sequence $x[n]$ is given as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Now, $X(z)$ is expanded in the following form

$$X(z) = .. + x[-2]\, z^{2} + x[-1]\, z^{1} + x[0] + x[1]\, z^{-1} + x[2]\, z^{-2} + ...$$

To obtain inverse z-transform(i.e.$x[n]$), represent the given $X(z)$ in the form of above power series. Then by comparing we can get

$$x[n] = \{...x[-2],\, x[-1],\, x[0],\, x[1],\, x[2],...\}$$

► EXAMPLE

The time sequence $x[n]$, corresponding to z-transform $X(z) = (1+z^{-1})^3$, $|z| > 0$ is

(A) $\{\underset{\uparrow}{3},3,1,1\}$

(B) $\{\underset{\uparrow}{1},3,3,1\}$

(C) $\{1,3,3,\underset{\uparrow}{1}\}$

(D) $\{1,\underset{\uparrow}{3},3,1\}$

SOLUTION :

Given

$$X(z) = (1+z^{-1})^3 = 1 + 3z^{-1} + 3z^{-2} + z^{-3}$$

From the defination of z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=0}^{3} x[n]\, z^{-n}$$

$$X(z) = x[0]\, z^{0} + x[1]\, z^{-1} + x[2]\, z^{-2} + x[3]\, z^{-3}$$

By comparing

$$x[0] = 1,\; x[1] = 3,\; x[2] = 3,\; x[3] = 1$$

Hence (B) is correct option.

6.5 PROPERTIES OF Z-TRANSFORM

The unilateral and bilateral z-transforms possess a set of properties, which are useful in the analysis of DT signals and systems. The proofs of properties are given for bilateral transform only and can be obtained in a similar way for the unilateral transform.

6.5.1 Linearity

Let $x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)$, with ROC: R_1

and $x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z)$, with ROC: R_2

then, $ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z)$,

with ROC: at least $R_1 \cap R_2$

for both unilateral and bilateral z-transform.

Proof :

The z-transform of signal $\{ax_1[n] + bx_2[n]\}$ is given by equation (6.1.1) as follows

$$\mathcal{Z}\{ax_1[n] + bx_2[n]\} = \sum_{n=-\infty}^{\infty}\{ax_1[n] + bx_2[n]\}z^{-n}$$

$$= a\sum_{n=-\infty}^{\infty}x_1[n]z^{-n} + b\sum_{n=-\infty}^{\infty}x_2[n]z^{-n}$$

$$= aX_1(z) + bX_2(z)$$

Hence, $ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z)$

Like Laplace transform, the linearity property of z transform states that, the linear combination of DT sequences in the time domain is equivalent to linear combination of their z transform.

ROC : Since, the z-transform $X_1(z)$ is finite within the specified ROC, R_1. Similarly, $X_2(z)$ is finite within its ROC, R_2. Therefore, the linear combination $aX_1(z) + bX_2(z)$ should be finite at least within region $R_1 \cap R_2$.

In certain cases, due to the interaction between $x_1[n]$ and $x_2[n]$, which may lead to cancellation of certain terms, the overall ROC may be larger than the intersection of the two regions. On the other hand, if there is no common region between R_1 and R_2, the z-transform of $ax_1[n] + bx_2[n]$ does not exist.

► E X A M P L E

The z-transform of the sequence

$$x[n] = 2^{n+1}u[n] + 3^{n+1}u[-n-1] \text{ is}$$

(A) $\dfrac{5 + 12z^{-1}}{1 - 5z^{-1} + 6z^{-2}}$ (B) $\dfrac{z^{-1}}{1 - 5z^{-1} + 6z^{-2}}$

(C) $\dfrac{5}{1 - 5z^{-1} + 6z^{-2}}$ (D) $\dfrac{-1}{1 - 5z^{-1} + 6z^{-2}}$

SOLUTION :

$$x[n] = 2(2^n u[n]) + 3(3^n u[-n-1])$$

$$x[n] = 2x_1[n] + 3x_2[n]$$

From table 6.1, we have standard transformation

$$x_1[n] = 2^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-2z^{-1}} = X_1(z)$$

$$x_2[n] = 3^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{-1}{1-3z^{-1}} = X_2(z)$$

From the linearity property of z-transform

$$2x_1[n] + 3x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} 2X_1(z) + 3X_2(z)$$

$$\overset{\mathcal{Z}}{\longleftrightarrow} \frac{2}{1-2z^{-1}} - \frac{3}{1-3z^{-1}}$$

$$\overset{\mathcal{Z}}{\longleftrightarrow} \frac{2-6z^{-1}-3+6z^{-1}}{(1-2z^{-1})(1-3z^{-1})}$$

$$\overset{\mathcal{Z}}{\longleftrightarrow} \frac{-1}{1-5z^{-1}+6z^{-2}}$$

Hence (D) is correct option.

6.5.2 Time shifting

For the bilateral z-transform

If $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, with ROC R_x

then $x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z)$,

and $x[n+n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{n_0}X(z)$,

with ROC : R_x except for the possible deletion or addition of $z = 0$ or $z = \infty$.

Proof :

The bilateral z-transform of signal $x[n-n_0]$ is given by equation (6.1.1) as follows

$$\mathcal{Z}\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0]\, z^{-n}$$

Substituting $n - n_0 = \alpha$ on RHS, we get

$$\mathcal{Z}\{x[n-n_0]\} = \sum_{\alpha=-\infty}^{\infty} x[\alpha]\, z^{-(n_0+\alpha)}$$

$$= \sum_{\alpha=-\infty}^{\infty} x[\alpha]\, z^{-n_0} z^{-\alpha}$$

$$= z^{-n_0} \sum_{\alpha=-\infty}^{\infty} x[\alpha]\, z^{-\alpha}$$

$$\mathcal{Z}\{x[n-n_0]\} = z^{-n_0}X[z]$$

Similarly we can prove

$$\mathcal{Z}\{x[n+n_0]\} = z^{n_0}X[z]$$

ROC : The ROC of shifted signals is altered because of the terms z^{n_0} or z^{-n_0}, which affects the roots of the denominator in $X(z)$.

For the unilateral z-transform

If $\quad x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad$ with ROC R_x

then $\quad x[n-n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0}\left(X(z) + \sum_{m=1}^{n_0} x[-m]\,z^{m}\right),$

and $\quad x[n+n_0] \xleftrightarrow{\mathcal{Z}} z^{n_0}\left(X(z) - \sum_{m=0}^{n_0-1} x[m]\,z^{-m}\right),$

with ROC : R_x except for the possible deletion or addition of $z=0$ or $z=\infty$.

Proof :

The unilateral z-transform of signal $x[n-n_0]$ is given by equation (6.1.2) as follows

$$\mathcal{Z}\{x[n-n_0]\} = \sum_{n=0}^{\infty} x[n-n_0]\,z^{-n}$$

Multiplying RHS by z^{n_0} and z^{-n_0}

$$\mathcal{Z}\{x[n-n_0]\} = \sum_{n=0}^{\infty} x[n-n_0]\,z^{-n}z^{n_0}z^{-n_0}$$

$$= z^{-n_0}\sum_{n=0}^{\infty} x[n-n_0]\,z^{-(n-n_0)}$$

Substituting $\quad n-n_0 = \alpha$

Limits; when $n \to 0,\ \alpha \to -n_0$

when $n \to +\infty,\ \alpha \to +\infty$

Now, $\quad \mathcal{Z}\{x[n-n_0]\} = z^{-n_0}\sum_{\alpha=-n_0}^{\infty} x[\alpha]\,z^{-\alpha}$

$$= z^{-n_0}\sum_{\alpha=-n_0}^{-1} x[\alpha]\,z^{-\alpha} + z^{-n_0}\sum_{\alpha=0}^{\infty} x[\alpha]\,z^{-\alpha}$$

or, $\quad \mathcal{Z}\{x[n-n_0]\} = z^{-n_0}\sum_{\alpha=0}^{\infty} x[\alpha]\,z^{-\alpha} + z^{-n_0}\sum_{\alpha=-n_0}^{-1} x[\alpha]\,z^{-\alpha}$

or, $$\mathcal{Z}\{x[n-n_0]\} = z^{-n_0}\sum_{\alpha=0}^{\infty} x[\alpha]\,z^{-\alpha} + z^{-n_0}\sum_{\alpha=1}^{n_0} x[-\alpha]\,z^{\alpha}$$

by changing the variables as $\alpha \to n$ and $\alpha \to m$ in first and second summation respectively

$$\mathcal{Z}\{x[n-n_0]\} = z^{-n_0}\sum_{n=0}^{\infty} x[n]\,z^{-n} + z^{-n_0}\sum_{m=1}^{n_0} x[-m]\,z^{m}$$

$$= z^{-n_0}X[z] + z^{-n_0}\sum_{m=1}^{n_0} x[-m]\,z^{m}$$

In similar way, we can also prove that

$$x[n+n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{n_0}\left(X(z) - \sum_{m=0}^{n_0-1} x[m]\,z^{-m}\right)$$

► E X A M P L E

Let $x[n]$ be a non-causal sequence with initial values $x[-1] = 2,\ x[-2] = 3$. If $X(z)$ represents the z-transform of $x[n]$ then z-transform of sequence

$$y[n] = ((x[n] - 3x[n-1]) + 4x[n-2])\,u[n] \text{ will be}$$

(A) $X(z)\,[1 - 3z^{-1} + 4z^{-2}] + 6 + 8z^{-1}$

(B) $X(z)\,[1 + 5z^{-1} + 4z^{-2}]$

(C) $X(z)\,[1 + 5z^{-1} + 4z^{-2}] + 6$

(D) $X(z)\,[1 - 3z^{-1} + 4z^{-2}]$

SOLUTION :

$$u[n] = 1, \qquad n \ge 0$$

So $X(z)$ is unilateral z-transform of $x[n]$. For unilateral z -transform, we have time shifting property as

$$x[n-n_0]\,u[n] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}\left(X(z) + \sum_{m=1}^{n_0} x[-m]\,z^{m}\right)$$

Thus

$$x[n-1]\,u[n] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-1}\left(X(z) + \sum_{m=1}^{1} x[-m]\,z^{m}\right)$$

$$\longleftrightarrow z^{-1}(X(z) + x[-1]\,z)$$

$$\longleftrightarrow z^{-1}X(z) + 2$$

Similarly

$$x[n-2]\,u[n] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-2}\left(X(z) + \sum_{m=1}^{2} x[-m]\,z^{m}\right)$$

$$\overset{\mathcal{Z}}{\longleftrightarrow} z^{-2}(X(z) + x[-1]\,z + x[-2]\,z^{2})$$

$$\overset{\mathcal{Z}}{\longleftrightarrow} z^{-2}X(z) + 2z^{-1} + 3$$

So z-transform of $y[n]$

$$Y(z) = X(z) - 3[z^{-1}X(z) + 2] + 4[z^{-2}X(z) + 2z^{-1} + 3]$$
$$= X(z)[1 - 3z^{-1} + 4z^{-2}] + 6 + 8z^{-1}$$

Hence (A) is correct option.

▶ EXAMPLE

Let $X(z)$ be the bilateral z-transform of a sequence $x[n]$ given as

$$X(z) = \frac{1}{z^2 - 4}, \qquad \text{ROC} : |Z| < 2$$

The z-transform of signal $x[n-2]$ will be

(A) $\frac{z^2}{z^2 - 4}$ (B) $\frac{1}{(z-2)^2 - 4}$

(C) $\frac{z^{-2}}{z^2 - 4}$ (D) $\frac{1}{(z+2)^2 - 4}$

SOLUTION :

For bilateral z-transform time shifting property states that

If, $$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$
$$x[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z)$$

So $$x[n-2] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-2}X(z) = \frac{z^{-2}}{z^2 - 4}$$

Hence (C) is correct option.

6.5.3 Time Reversal

If $$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad \text{with ROC} : R_x$$

then $$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{1}{z}\right), \qquad \text{with ROC} : 1/R_x$$

For bilateral z-transform.

Time reversal property states that time reflection of a DT sequence in time domain is equivalent to replacing z by $1/z$ in its z-transform.

Proof :

The bilateral z-transform of signal $x[-n]$ is given by equation (6.1.1) as follows

$$\mathcal{Z}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]z^{-n}$$

Substituting $-n = k$ on the RHS, we get

$$\mathcal{Z}\{x[-n]\} = \sum_{k=\infty}^{-\infty} x[k]\, z^{k}$$
$$= \sum_{k=-\infty}^{\infty} x[k]\,(z^{-1})^{-k}$$
$$= X\left(\frac{1}{z}\right)$$

Hence, $x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{1}{z}\right)$

ROC : $z^{-1} \in R_x$ or $z \in 1/R_x$

► EXAMPLE

Let $\alpha^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} 1/(1-\alpha z^{-1})$, then what will be the z -transform of sequence $\alpha^{-n} u[-n]$?

(A) $\frac{1}{1-\alpha z}$ (B) $\frac{\alpha}{z-1}$

(C) $\frac{z}{z-\alpha}$ (D) $\frac{1}{z-\alpha}$

SOLUTION :

$$\alpha^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}$$

By time reversal property

$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z^{-1})$$

So $\alpha^{-n} u[-n] = \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\alpha(z^{-1})^{-1}} = \frac{1}{1-\alpha z}$

Hence (A) is correct option.

6.5.4 Differentiation in the z-domain

If $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, with ROC : R_x

then $nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}$, with ROC : R_x

For both unilateral and bilateral z-transforms.

Proof :

The bilateral z-transform of signal $x[n]$ is given by equation (6.1.1) as follows

This property states that multiplication of time sequence $x[n]$ with n corresponds to differentiation with respect to z and multiplication of result by $-z$ in the z -domain.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Differentiating both sides with respect to z gives

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n] \frac{dz^{-n}}{dz} = \sum_{n=-\infty}^{\infty} x[n]\,(-\, nz^{-n-1})$$

Multiplying both sides by $-z$, we obtain

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx[n]\, z^{-n}$$

Hence, $$nx[n] \overset{z}{\longleftrightarrow} -\, z\frac{dX(z)}{dz}$$

ROC : This operation does not affect the ROC.

▶ EXAMPLE

Which of the following corresponds to z-transform of the sequence $x[n] = (n+1)\, a^n\, u[n]$?

(A) $\dfrac{az^{-1}}{(1 - az^{-1})^2}$

(B) $\dfrac{z^{-1}}{(1 - az^{-1})^2}$

(C) $\dfrac{1}{(1 - az^{-1})^2}$

(D) $\dfrac{(1 + az^{-1})}{(1 - az^{-1})}$

SOLUTION :

$$x[n] = na^n\, u[n] + a^n\, u[n]$$

We know that

$$a^n\, u[n] \overset{z}{\longleftrightarrow} \frac{1}{(1 - az^{-1})}$$

Using property of z-domain differentiation

$$na^n\, u[n] \overset{z}{\longleftrightarrow} -\, z\frac{d}{dz}\left[\frac{1}{(1 - az^{-1})}\right]$$

$$\overset{z}{\longleftrightarrow} \frac{az^{-1}}{(1 - az^{-1})^2}$$

Using Linearity property

$$a^n\, u[n] + na^n\, u[n] \overset{z}{\longleftrightarrow} \frac{1}{(1 - az^{-1})} + \frac{az^{-1}}{(1 - az^{-1})^2}$$

$$\overset{z}{\longleftrightarrow} \frac{1}{(1 - az^{-1})^2}$$

Hence (C) is correct option.

6.5.5 Scaling in z-domain

If $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, with ROC : R_x

then $a^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{z}{a}\right)$, with ROC : $|a| R_x$

For both unilateral and bilateral transform.

Multiplication of a time sequence with an exponential sequence a^n corresponds to scaling in z-domain by a factor of a.

Proof :

The bilateral z-transform of signal $x[n]$ is given by equation (6.1.1) as

$$\mathcal{Z}\{a^n x[n]\} = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] [a^{-1} z]^{-n}$$

$$a^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{z}{a}\right)$$

ROC : If z is a point in the ROC of $X(z)$ then the point $|a| z$ is in the ROC of $X(z/a)$.

► EXAMPLE

If the z-transform of unit step sequence is given as $u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}}$, then the z-transform of sequence $\left(\frac{1}{3}\right)^n u[n]$ will be

(A) $\frac{3}{(1 - z^{-1})}$

(B) $\frac{1}{3(1 - z^{-1})}$

(C) $\frac{1}{\left(1 - \frac{1}{3} z^{-1}\right)}$

(D) $\frac{1}{(1 - 3z^{-1})}$

SOLUTION :

If $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$

$$a^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{z}{a}\right)$$

[Property of scaling in z-domain]

$$\left(\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \left(\frac{z}{1/3}\right)^{-1}} = \frac{1}{\left(1 - \frac{1}{3} z^{-1}\right)}$$

Hence (C) is correct option.

6.5.6 Time Scaling

As we discussed in Chapter 2, there are two types of scaling in the DT domain decimation(compression) and interpolation(expansion).

Time Compression

Since the decimation (compression) of DT signals is an irreversible process (because some data may lost), therefore the z-transform of $x[n]$ and its decimated sequence $y[n] = x[an]$ not be related to each other.

Time Expansion

In the discrete time domain, time expansion of sequence $x[n]$ is defined as

$$x_k[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of integer } k \\ 0 & \text{otherwise} \end{cases} \tag{6.5.1}$$

Time-scaling property of z-transform is derived only for time expansion which is given as

Time expansion of a DT sequence by a factor of k corresponds to replacing z as z^k in its z-transform.

If $\quad x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad$ with ROC : R_x

then $\quad x_k[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_k(z) = X(z^k), \qquad$ with ROC : $(R_x)^{1/k}$

For both the unilateral and bilateral z-transform.

Proof :

The unilateral z-transform of expanded sequence $x_k[n]$ is given by

$$\begin{aligned} \mathcal{Z}\{x_k[n]\} &= \sum_{n=0}^{\infty} x_k[n]\, z^{-n} \\ &= x_k[0] + x_k[1]\, z^{-1} + ... + x_k[k]\, z^{-k} \\ &\quad + x_k[k+1]\, z^{-(k+1)} + ... x_k[2k]\, z^{-2k} + ... \end{aligned}$$

Since the expanded sequence $x_k[n]$ is zero everywhere except when n is a multiple of k. This reduces the above transform as follows

$$\mathcal{Z}\{x_k[n]\} = x_k[0] + x_k[k]\, z^{-k} + x_k[2k]\, z^{-2k} + x_k[3k]\, z^{-3k} + ...$$

As defined in equation 6.5.1, interpolated sequence is

$$x_k[n] = x[n/k]$$

$n = 0 \qquad x_k[0] = x[0],$

$n = k \qquad x_k[k] = x[1]$

$n = 2k \qquad x_k[2k] = x[2]$

Thus, we can write

$$\mathcal{Z}\{x_k[n]\} = x[0] + x[1]\,z^{-k} + x[2]\,z^{-2k} + x[3]\,z^{-3k} + ...$$

$$= \sum_{n=0}^{\infty} x[n]\,(z^k)^{-n} = X(z^k)$$

► **E X A M P L E**

Let $X(z)$ be z-transform of a DT sequence $x[n] = (-0.5)^n u[n]$ .Consider another signal $y[n]$ and its z-transform $1/(z)$ given as

$$Y(z) = X(z^2)$$

What is the value of $y[n]$ at $n = 4$?

(A) 2 (B) 4

(C) 1/2 (D) 1/4

SOLUTION :

We know that

if $\qquad x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$

$$x\left[\frac{n}{2}\right] \overset{\mathcal{Z}}{\longleftrightarrow} X(z^2) \text{ (time expansion property)}$$

So $\qquad y[n] = x\left[\frac{n}{2}\right]$

$$y[n] = \begin{cases} (-0.5)^{n/2}, & n = 0, 2, 4, 6... \\ 0, & \text{otherwise} \end{cases}$$

So $\qquad y[4] = (-0.5)^2 = \frac{1}{4}$

Hence (D) is correct option.

6.5.7 Time Differencing

If $\qquad x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad$ with ROC : R_x

then $\quad x[n] - x[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} (1 - z^{-1})\,X(z),$

with the ROC : R_x except for the possible deletion of $z = 0$.

For both unilateral and bilateral transform.

Proof :

The z-transform of $x[n]-x[n-1]$ is given by equation (6.1.1) as follows

$$\mathcal{Z}\{x[n]-x[n-1]\} = \sum_{n=-\infty}^{\infty}\{x[n]-x[n-1]\}\, z^{-n}$$
$$= \sum_{n=-\infty}^{\infty}x[n]\, z^{-n} - \sum_{n=-\infty}^{\infty}x[n-1]\, z^{-n}$$

In the second summation, substituting $n-1=r$

$$\mathcal{Z}\{x[n]-x[n-1]\} = \sum_{n=-\infty}^{\infty}x[n]\, z^{-n} - \sum_{r=-\infty}^{\infty}x[r]\, z^{-(r+1)}$$
$$= \sum_{n=-\infty}^{\infty}x[n]\, z^{-n} - z^{-1}\sum_{r=-\infty}^{\infty}x[r]\, z^{-r}$$
$$= X(z) - z^{-1}X(z)$$

Hence,

$$x[n]-x[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} (1-z^{-1})\,X(z)$$

▶ EXAMPLE

If the z-transform of unit-step sequence is given as $u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}$, then the z-transform of $au[n]-bu[n-1]$ will be

(A) $\frac{(b-az^{-1})}{(1-z^{-1})}$

(B) $\frac{a}{1-bz^{-1}}$

(C) $\frac{(a-bz^{-1})}{(1-z^{-1})}$

(D) $\frac{b}{(1-az^{-1})}$

SOLUTION :

Let $x[n]=u[n],\ X(z)=\frac{1}{(1-z^{-1})}$

From time differencing property

$$ax[n]-bx[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} (a-bz^{-1})\,X(z)$$
$$au[n]-bu[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} (a-bz^{-1})\left(\frac{1}{1-z^{-1}}\right)$$

Hence (C) is correct option.

6.5.8 Time Convolution

Let $x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)$, ROC : R_1

and $x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z)$, ROC : R_2

then the convolution property states that

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)\,X_2(z),$$

ROC : at least $R_1 \cap R_2$

For both unilateral and bilateral z-transforms.

Time convolution property states that convolution of two sequence in time domain corresponds to multiplication in z-domain.

Proof :

As discussed in chapter 4, the convolution of two sequences is given by

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]$$

Taking the z-transform of both sides gives

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]\,z^{-n}$$

By interchanging the order of the two summations, we get

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]\,z^{-n}$$

Substituting $n-k=\alpha$ in the second summation

$$x[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} x_1[k] \sum_{\alpha=-\infty}^{\infty} x_2[\alpha]\,z^{-(\alpha+k)}$$

or
$$x[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} \left(\sum_{k=-\infty}^{\infty} x_1[k]\,z^{-k}\right)\left(\sum_{\alpha=-\infty}^{\infty} x_2[\alpha]\,z^{-\alpha}\right)$$

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)\,X_2(z)$$

► E X A M P L E

Consider a sequence $x[n] = x_1[n] * x_2[n]$ and its z-transform $X(z)$. It is given that

$$x_1[n] = \{1, 2, 2\}$$

and
$$x_2[n] = \begin{cases} 1, & 0 \le n \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

then $X(z)\big|_{z=1}$ will be

(A) 8 (B) 15

(C) 7 (D) 4

SOLUTION :

$$x[n] = x_1[n] * x_2[n]$$

Using convolution property

$$X(z) = X_1(z)\,X_2(z)$$

$$x_1[n] = \{1,2,2\}$$

$$X_1(z) = \sum_{n=0}^{2} x_1[n]\,z^{-n}$$

$$= 1 + 2z^{-1} + 2z^{-2}$$

$$x_2[n] = \{1,1,1\}$$

$$X_2(z) = \sum_{n=0}^{2} x_2[n]\,z^{-n}$$

$$= 1 + z^{-1} + z^{-2}$$

$$X(z) = (1 + 2z^{-1} + 2z^{-2})(1 + z^{-1} + z^{-2})$$

$$= (1 + z^{-1} + z^{-2} + 2z^{-1} + 2z^{-2} + 2z^{-3} + 2z^{-2} + 2z^{-3} + 2z^{-4})$$

$$= 1 + 3z^{-1} + 5z^{-2} + 4z^{-3} + 2z^{-4}$$

$$= 1 + 3 + 5 + 4 + 2$$

$$= 15$$

Hence (B) is correct option.

6.5.9 Conjugation Property

If $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, with ROC : R_x

then $x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$, with ROC : R_x

If $x[n]$ is real, then

$$X(z) = X^*(z^*)$$

Proof :

The z-transform of signal $x^*[n]$ is given by equation (6.1.1) as follows

$$\mathcal{Z}\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n]\,z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[x[n]\,(z^*)^{-n}\right]^* \qquad (6.5.2)$$

Let z-transform of $x[n]$ is $X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\,z^{-n}$$

by taking complex conjugate on both sides of above equation

$$X^*(z) = \sum_{n=-\infty}^{\infty} [x[n]\,z^{-n}]^*$$

Replacing $z \to z^*$, we will get

$$X^*(z^*) = \sum_{n=-\infty}^{\infty} \left[x[n]\,(z^*)^{-n}\right]^* \qquad (6.5.3)$$

Comparing equation (6.5.2) and (6.5.3)

$$\mathcal{Z}\{x^*[n]\} = X^*(z^*) \qquad (6.5.4)$$

For real $x[n]$, $\quad x^*[n] = x[n]$, so

$$\mathcal{Z}\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x[n]\,z^{-n} = X(z) \qquad (6.5.5)$$

Comparing equation (6.5.4) and (6.5.5)

$$X(z) = X^*(z^*)$$

6.5.10 Initial Value Theorem

If $\quad x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad$ with ROC : R_x
then initial-value theorem states that,

$$x[0] = \lim_{z \to \infty} X(z)$$

The initial-value theorem is valid only for the unilateral Lapalce transform

Proof :

For a causal signal $x[n]$

$$X(z) = \sum_{n=0}^{\infty} x[n]\,z^{-n}$$

$$= x[0] + x[1]\,z^{-1} + x[2]\,z^{-2} + \ldots$$

Taking limit as $z \to \infty$ on both sides we get

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} (x[0] + x[1]\,z^{-1} + x[2]\,z^{-2} + \ldots)$$
$$= x[0]$$
$$x[0] = \lim_{z \to \infty} X(z)$$

► EXAMPLE

The z −transform of a causal system is given as

$$X(z) = \frac{2 - 1.5z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

The value of x[0] is

(A) −1.5 (B) 2

(C) 1.5 (D) 0

SOLUTION :

Causal signal $x[0] = \lim_{z \to \infty} X(z) = 2$

Hence (B) is correct option.

6.5.11 Final Value Theorem

If $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, with ROC : R_x

then final-value theorem states that

$$x[\infty] = \lim_{z \to 1} (z-1)\,X(z)$$

The final-value theorem, can be applicable with either the unilateral or bilateral z-transform.

Final value theorem is applicable if $X(z)$ has no poles outside the unit circle.

Proof :

$$\mathcal{Z}\{x[n+1]\} - \mathcal{Z}\{x[n]\} = \lim_{k \to \infty} \sum_{n=0}^{k} \{x[n+1] - x[n]\}\,z^{-n} \quad (6.5.6)$$

From the time shifting property of unilateral z-transform discussed in section 6.5.2

$$x[n+n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{n_0}\left(X(z) - \sum_{m=0}^{n_0-1} x[m]\,z^{-m}\right)$$

For $n_0 = 1$

$$x[n+1] \overset{\mathcal{Z}}{\longleftrightarrow} z\left(X(z) - \sum_{m=0}^{0} x[m]\, z^{-m}\right)$$

$$x[n+1] \overset{\mathcal{Z}}{\longleftrightarrow} z(X(z) - x[0])$$

Put above transformation in the equation (6.5.6)

$$zX[z] - zx[0] - X[z] = \lim_{k\to\infty} \sum_{n=0}^{k} (x[n+1] - x[n])\, z^{-n}$$

$$(z-1)\, X[z] - zx[0] = \lim_{k\to\infty} \sum_{n=0}^{k} (x[n+1] - x[n])\, z^{-n}$$

Taking limit as $z \to 1$ on both sides we get

$$\lim_{z\to 1} (z-1)\, X[z] - x[0] = \lim_{k\to\infty} \sum_{n=0}^{k} x[n+1] - x[n]$$

$$\lim_{z\to 1} (z-1)\, X[z] - x[0]$$
$$= \lim_{k\to\infty} \{(x[1] - x[0]) + (x[2] - x[1]) + (x[3] - x[2]) + \dots$$
$$\dots + (x[k+1] - x[k])$$

$$\lim_{z\to 1} (z-1)\, X[z] - x[0] = x[\infty] - x[0]$$

Hence, $$x[\infty] = \lim_{z\to 1} (z-1)\, X(z)$$

► EXAMPLE

Given the $z-$transforms

$$X(z) = \frac{z(8z-7)}{4z^2 - 7z + 3}$$

The limit of $x[\infty]$ is

(A) 1 (B) 2

(C) ∞ (D) 0

SOLUTION :

The function has poles at $z = 1, \frac{3}{4}$. Thus final value theorem applies.

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1} (z-1)\, X(z)$$

$$= (z-1) \frac{z(2z - \frac{7}{4})}{(z-1)(z - \frac{3}{4})} = 1$$

Hence (A) is correct option.

Summary :

Let,

$$x[n] \xleftrightarrow{z} X(z), \qquad \text{with ROC } R_x$$
$$x_1[n] \xleftrightarrow{z} X_1(z), \qquad \text{with ROC } R_1$$
$$x_2[n] \xleftrightarrow{z} X_2(z), \qquad \text{with ROC } R_2$$

The properties of z-transforms are summarized in the following table.

TABLE 6.2 Properties of z-transform

Properties	**Time domain**	**z-transform**	**ROC**
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	at least $R_1 \cap R_2$
Time shifting (bilateral or non-causal)	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x except for the possible deletion or addition of $z = 0$ or $z = \infty$
	$x[n+n_0]$	$z^{n_0}X(z)$	
Time shifting (unilateral or causal)	$x[n-n_0]$	$z^{-n_0}\left(X(z) + \sum_{m=1}^{n_0} x[-m]\,z^{m}\right)$	R_x except for the possible deletion or addition of $z = 0$ or $z = \infty$
	$x[n+n_0]$	$z^{n_0}\left(X(z) - \sum_{m=0}^{n_0-1} x[m]\,z^{-m}\right)$	
Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$	$1/R_x$
Differentiation in z domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
Scaling in z domain	$a^n x[n]$	$X\left(\frac{z}{a}\right)$	$\lvert a \rvert R_x$
Time scaling(expansion)	$x_k[n] = x[n/k]$	$X(z^k)$	$(R_x)^{1/k}$
Time differencing	$x[n] - x[n-1]$	$(1-z^{-1})X(z)$	R_x, except for the possible deletion of the origin

Time convolution	$x_1[n] * x_2[n]$	$X_1(z)\,X_2(z)$	at least $R_1 \cap R_2$
Conjugations	$x^*[n]$	$X^*(z^*)$	R_x
Initial-value theorem		$x[0] = \lim_{z \to \infty} X(z)$	provided $x[n] = 0$ for $n < 0$
Final-value theorem		$x[\infty] = \lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1)\,X(z)$	provided $x[\infty]$ exists

6.6 ANALYSIS OF DISCRETE LTI SYSTEMS USING z-TRANSFORM

The z-transform is very useful tool in the analysis of discrete LTI system. As the Laplace transform is used in solving differential equations which describe continuous LTI systems, the z-transform is used to solve difference equation which describe the discrete LTI systems.

Similar to Laplace transform, for CT domain, the z-transform gives transfer function of the LTI discrete systems which is the ratio of the z-transform of the output variable to the z-transform of the input variable.
These applications are discussed as follows

6.6.1 Response of LTI Continuous Time System

As discussed in chapter 4 (section 4.8), a discrete-time LTI system is always described by a linear constant coefficient difference equation given as follows

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

$$a_N\, y[n-N] + a_{N-1} y[n-(N-1)] + ... + a_1 y[n-1] + a_0\, y[n]$$
$$= b_M\, x[n-M] + b_{M-1} x[n-(M-1)] + ... + b_1 x[n-1] + b_0 x[n] \quad (6.6.1)$$

where, N is order of the system.

The time-shift property of z-transform $x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z)$, is used to solve the above difference equation which converts it into an algebraic equation. By taking z-transform of above equation

$$a_N\, z^{-N}\, Y(z) + a_{N-1} z^{-(N-1)}\, Y(z)] + ... + a_1 z^{-1} + a_0\, Y(z)$$
$$= b_M\, z^{-M}\, X(z) + b_{M-1} z^{-(M-1)} X(z) + ... + b_1 z^{-1} X(x) + b_0 X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{b_M z^{-N} + b_{M-1} z^{M-1} + ... + b_1 + b_0}{a_N z^N + a_{N-1} z^{N-1} + ... + a_1 + a_0}$$

this equation can be solved for $Y(z)$ to find the response $y[n]$.The solution or total response $y[n]$ consists of two parts as discussed below.

1. Zero-input Response or Free Response or Natural Response

The zero input response $y_{zi}[n]$ is mainly due to initial output in the system. The zero-input response is obtained from system equation (6.6.1) when input $x[n] = 0$.

By substituting $x[n] = 0$ and $y[n] = y_{zi}[n]$ in equation (6.6.1), we get

$$a_N y[n-N] + a_{N-1} y[n-(N-1)] + ... + a_1 y[n-1] + a_0 y[n] = 0$$

On taking z-transform of the above equation with given initial conditions, we can form an equation for $Y_{zi}(z)$. The zero-input response $y_{zi}[n]$ is given by inverse z-transform of $Y_{zi}(z)$.

The zero input response is also called the natural response of the system and it is denoted as $y_N[n]$.

2. Zero-State Response or Forced Response

The zero-state response $y_{zs}[n]$ is the response of the system due to input signal and with zero initial conditions. The zero-state response is obtained from the difference equation (6.6.1) governing the system for specific input signal $x[n]$ for $n \geq 0$ and with zero initial conditions.

On substituting $y[n] = y_{zs}[n]$ in equation (6.6.1) we get,

$$a_N y_{zs}[n-N] + a_{N-1} y_{zs}[n-(N-1)] + ... + a_1 y_{zs}[n-1] + a_0 y_{zs}[n]$$
$$= b_M x[n-M] + b_{M-1} x[n-(M-1)] + ... + b_1 x[n-1] + b_0 x[n]$$

By taking z-transform of the above equation with zero initial conditions for output (i.e., $y[-1] = y[-2]... = 0$ we can form an equation for $Y_{zs}(z)$.

The zero-state response $y_{zs}[n]$ is given by inverse z-transform of $Y_{zs}(z)$.

The zero state response is also called the forced response of the system and it is denoted as $y_F[n]$.

Total Response

The total response $y[n]$ is the response of the system due to input signal and initial output. The total response can be obtained in following two ways :

By taking z-transform of equation (6.6.1) with non-zero initial conditions for both input and output, and then substituting for $X(z)$ we can form an equation for

$Y(z)$. The total response $y[n]$ is given by inverse Laplace transform of $Y(s)$.

Alternatively, that total response $y[n]$ is given by sum of zero-input response $y_{zi}[n]$ and zero-state response $y_{zs}[n]$.
$\therefore$ Total response,

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

► EXAMPLE

A discrete time system has the following input-output relationship

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

If an input $x[n] = u[n]$ is applied to the system, then its zero state response will be

(A) $\left[\frac{1}{2} - (2)^n\right]u[n]$

(B) $\left[2 - \left(\frac{1}{2}\right)^n\right]u[n]$

(C) $\left[\frac{1}{2} - \left(\frac{1}{2}\right)^n\right]u[n]$

(D) $[2 - (2)^n]\,u[n]$

SOLUTION :

zero state response refers to response of the system with zero initial conditions.
By taking z-transform

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z)$$

$$Y(z) = \left(\frac{z}{z-0.5}\right)X(z)$$

For an input $x[n] = u[n],\ X(z) = \dfrac{z}{z-1}$

so, $$Y(z) = \frac{z}{(z-0.5)}\frac{z}{(z-1)}$$

$$Y(z) = \frac{z^2}{(z-1)(z-0.5)}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

By partial fraction

$$\frac{Y(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$Y(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

By taking inverse z-transform

$$y[n] = 2u[n] - (0.5)^n u[n]$$

Hence (B) is correct option.

6.6.2 Impulse Response and Transfer Function

System function or transfer function is defined as the ratio of the z-transform of the output $y[n]$ and the input $x[n]$ with zero initial conditions.

Let $x[n] \xleftrightarrow{\mathcal{Z}} X(z)$ is the input and $y[n] \xleftrightarrow{\mathcal{L}} Y(z)$ is the output of an LTI discrete time system having impulse response $h(n) \xleftrightarrow{\mathcal{L}} H(z)$. The response $y[n]$ of the discrete time system is given by convolution sum of input and impulse response as

$$y[n] = x[n] * h[n]$$

By applying convolution property of z-transform we obtain

$$Y(z) = X(z)\,H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

where, $H(z)$ is defined as the transfer function of the system. It is the z-transform of the impulse response.

Alternatively we can say that the inverse z-transform of transfer function is the impulse response of the system.

Impulse response

$$h[n] = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{Y(z)}{X(z)}\right\}$$

► EXAMPLE

A system is described by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = 2x[n-1]$$

The impulse response of the system is

(A) $\frac{1}{2^{n-2}}u[n-1]$ (B) $\frac{1}{2^{n-2}}u[n+1]$

(C) $\frac{1}{2^{n-2}}u[n-2]$ (D) $\frac{-1}{2^{n-2}}u[n-2]$

SOLUTION :

$$Y(z)\left[1-\frac{z^{-1}}{2}\right]=2z^{-1}X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{2z^{-1}}{1-\frac{z^{-1}}{2}}$$

$$\Rightarrow \qquad h[n]=2\left(\frac{1}{2}\right)^{n-1}u[n-1]$$

Hence (A) is correct option.

6.7 Stability & Causality of LTI Discrete Systems Using z-Transform

z-transform is also used in characterization of LTI discrete systems. In this section, we derive a z-domain condition to check the stability and causality of a system directly from its z-transfer function.

6.7.1 Causality

A linear time-invariant discrete time system is said to be causal if the impulse response $h[n]=0$, for $n<0$ and it is therefore right-sided. The ROC of such a system $H(z)$ is the exterior of a circle. If $H(z)$ is rational then the system is said to be causal if

(A) The ROC is the exterior of a circle outside the outermost pole ; and
(B) The degree of the numerator polynomial of $H(z)$ should be less than or equal to the degree of the denominator polynomial.

6.7.2 Stability

An LTI discrete-time system is said to be BIBO stable if the impulse response $h[n]$ is summable. That is

$$\sum_{n=-\infty}^{\infty}\left|h[n]\right|<\infty$$

z-transform of $h[n]$ is given as

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

Let $z = e^{j\Omega}$ (which describes a unit circle in the z-plane), then

$$\left| H[e^{j\Omega}] \right| = \left| \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\Omega n} \right|$$
$$\leq \sum_{n=-\infty}^{\infty} \left| h[n]\, e^{-j\Omega n} \right|$$
$$= \sum_{n=-\infty}^{\infty} \left| h[n] \right| < \infty$$

which is the condition for the stability. Thus we can conclude that

> An LTI system is stable if the ROC of its system function $H(z)$ contains the unit circle $|z| = 1$

6.7.3 Stability & Causality

As we discussed previously, for a causal system with rational transfer function $H(z)$, the ROC is outside the outermost pole. For the BIBO stability the ROC should include the unit circle $|z| = 1$. Thus, for the system to be causal and stable theses two conditions are satisfied if all the poles are within the unit circle in the z-plane.

> An LTI discrete time system with the rational system function $H(z)$ is said to be both causal and stable if all the poles of $H(z)$ lies inside the unit circle.

▶ E X A M P L E

A Linear time-invariant system has the following system function

$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

Consider the following statements about the system

1. The system is stable if ROC : $|z| > \frac{1}{2}$

2. The system is causal if ROC : $|z| > \frac{1}{2}$

3. The system is stable if ROC : $\frac{1}{2} < |z| < 3$

4. The system is causal if ROC : $|z| > 3$

Which of the above statement is/are correct?

(A) 1 and 2 (B) 1 and 3

(C) 2 and 3 (D) 3 and 4

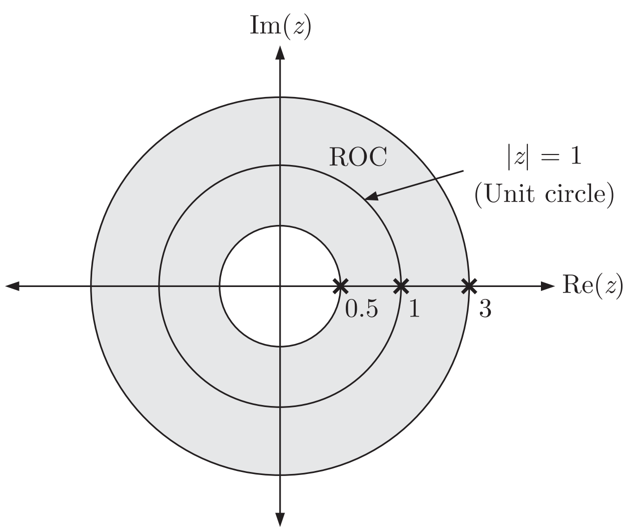
SOLUTION :

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

The system has poles at $z = \frac{1}{2}$ and $z = 3$

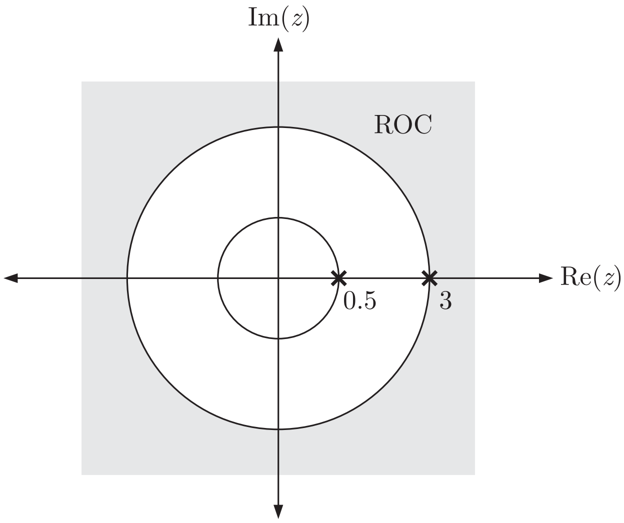
Stability:

An LTI system is stable only if ROC of $H(z)$ contains unit circle so ROC : $\frac{1}{2} < |z| < 3$



Causility:

For an LTI System to be causal the ROC must be exterior of a circle outside the outer most pole. Here outer most pole is $z = 3$. So for a causal system ROC : $|z| > 3$



Hence (D) is correct option.

▶ EXAMPLE

The transfer function of a discrete LTI system is given by

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}$$

Consider the following statements:

S_1 : The system is unstable and causal for ROC : $|z| > 2$

S_2 : The system is stable but not causal for ROC : $0.5 < |z| < 2$

S_3 : The system is neither stable nor causal for ROC : $|z| < 0.5$

Which of the above statement is true?

(A) All S_1, S_2 and S_3 are true

(B) Both S_1 and S_2 are true

(C) Both S_2 and S_3 are true

(D) Both S_1 and S_3 are true

SOLUTION :

The system has poles at $z = 1/2$ and $z = 2$. Now consider the different ROCs.

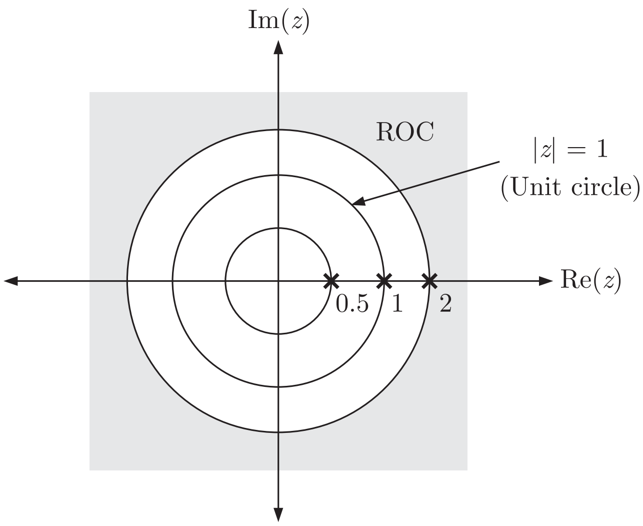
ROC : $|z| > 2$

Stability:

Since ROC does not contain unit circle. Hence the system is not stable.

Causality:

ROC is exterior to outer most pole $(z = 2)$ so the system is causal.



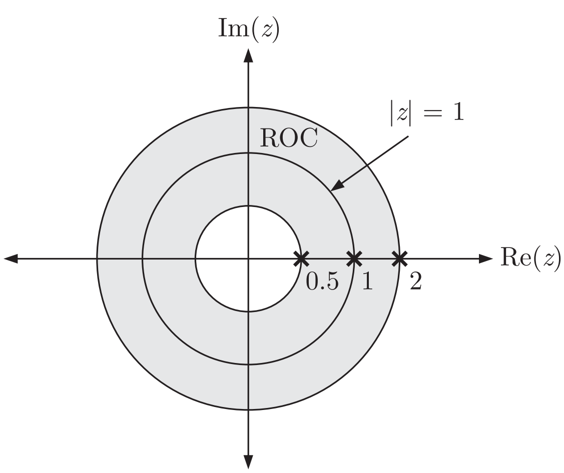
ROC : $0.5 < |z| < 2$

Stability:

ROC contains unit circle, so the system is stable.

Causility:

ROC is not exterior to outer most pole $(z = 2)$ so the system is not causal.



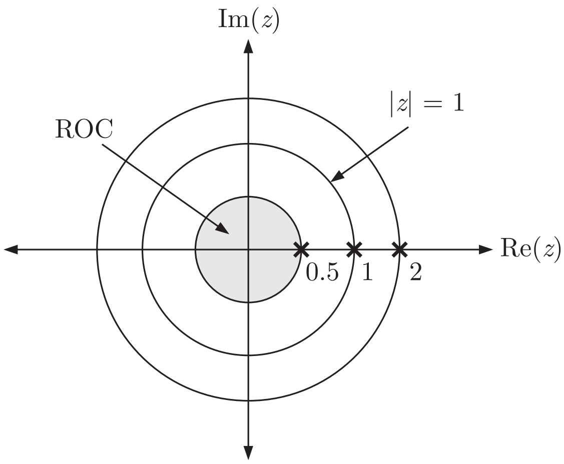
ROC : $|z| < 0.5$

Stability:

ROC does not contain unit circle so the system is unstable.

Causility:

ROC is not exterior to outer most pole ($z = 2$), hence it is not causal.



Hence (A) is correct option.

▶ EXAMPLE

The impulse response of a system is given by

$$h[n] = 10\left(\frac{-1}{2}\right)^n u[n] - 9\left(\frac{-1}{4}\right)u[n]$$

For this system two statement are

Statement (i) : System is causal and stable

Statement (ii) : Inverse system is causal and stable.

The correct option is

(A) (i) is true

(B) (ii) is true

(C) Both are true

(D) Both are false

SOLUTION :

$$H(z) = \frac{10}{1 + \frac{1}{2}z^{-1}} - \frac{9}{1 + \frac{1}{4}z^{-1}}$$

$$= \frac{1 - 2z^{-1}}{(1 + \frac{1}{2}z^{-1})\left(1 + \frac{1}{4}z^{-1}\right)}$$

Pole of this system are inside $|z| = 1$. So the system is stable and causal.

For the inverse system not all pole are inside $|z| = 1$. So inverse system is not stable and causal.

Hence (A) is correct option.

6.8 Block Diagram Representation

In z-domain, the input-output relation of an LTI discrete time system is represented by the transfer function $H(z)$,which is a rational function of z, as shown in equation

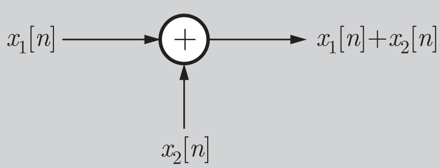
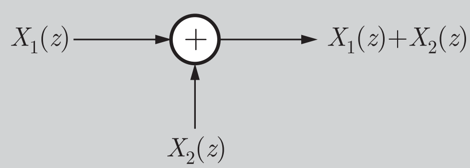
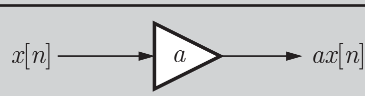
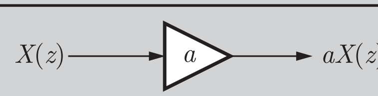
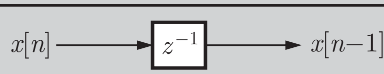
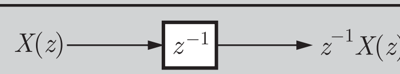
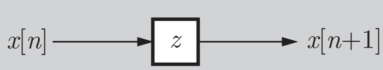
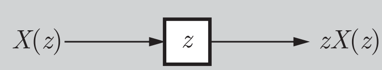
$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + ... + b_{M-1} z + b_M}{a_0 z^N + a_1 z^{N-1} + a_2 z^{N-2} + ... + a_{N-1} z + a_N}$$

where, $N =$ Order of the system, $M \leq N$ and $a_0 = 1$

The above transfer function is realized using unit delay elements, unit advance elements, adders and multipliers. Basic elements of block diagram with their z -domain representation is shown in table 6.3.

TABLE 6.3 : Basic Elements of Block Diagram

Elements of Block diagram	Time domain representation	s-domain representation
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

The different types of structures for realizing discrete time systems are same as we discussed for the continuous-time system in the previous chapter.

6.8.1 Direct Form I Realization

Consider the difference equation governing the discrete time system with $a_0 = 1$,

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + ... + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + ... + b_M x[n-M]$$

On taking $\mathcal{Z}$ transform of the above equation we get,

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - ... - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... + b_M z^{-M} X(z) \quad (6.8.1)$$

The above equation of $Y(z)$ can be directly represented by a block diagram as shown in figure 6.8.1a. This structure is called direct form-I structure. This structure uses separate delay elements for both input and output of the system. So, this realization uses more memory.

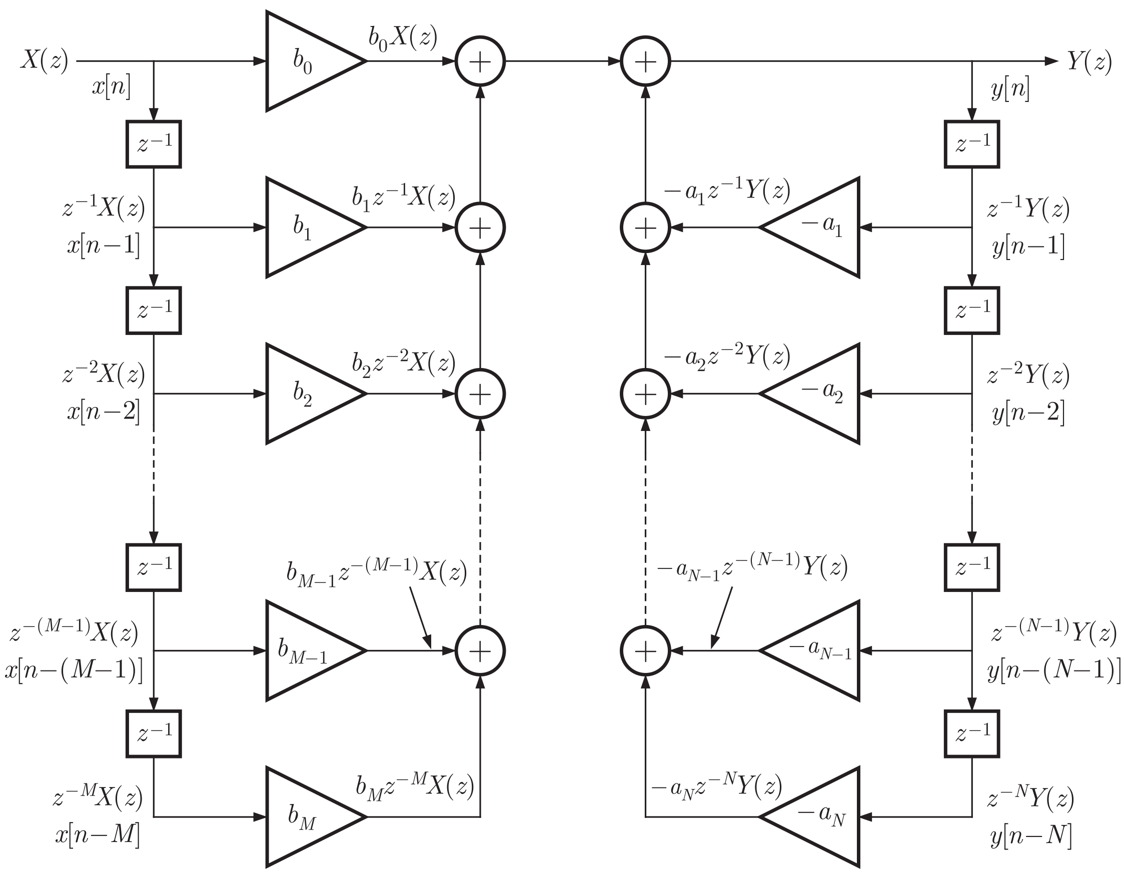


Fig 6.8.1a General structure of direct form-I realization

For example consider a discrete LTI system which has the following impulse response

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + 2z^{-2}}{1 + 4z^{-1} + 3z^{-2}}$$

$$Y(z) + 4z^{-1}Y(z) + 3z^{-2}Y(z) = 1X(z) + 2z^{-1}X(z) + 2z^{-2}X(z)$$

Comparing with standard form of equation (6.8.1), we get $a_1 = 4,\ a_2 = 3$ and $b_0 = 1,\ b_1 = 2,\ b_2 = 2$. Now put these values in general structure of Direct form-I realization we get

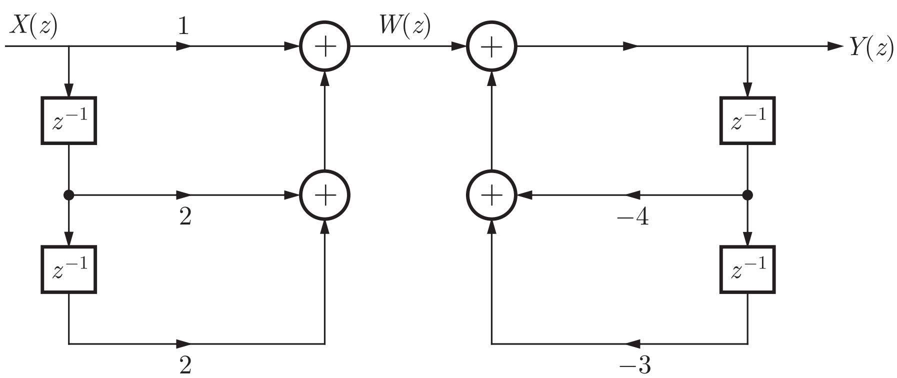


Fig 6.8.1b

6.8.2 Direct Form II Realization

Consider the general difference equation governing a discrete LTI system

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + ... + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + ... + b_M x[n-M]$$

On taking $\mathcal{Z}$ transform of the above equation we get,

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - ... - a_N z^{-N} Y(z) +$$
$$b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... + b_M z^{-M} X(z)$$

It can be simplified as,

$$Y(z)[1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}]$$
$$= X(z)[b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}]$$

Let, $\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)}$

where,

$$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}} \quad (6.8.2)$$

$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M} \quad (6.8.3)$$

Equation (6.8.2) can be simplified as,

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + ... + a_N z^{-N} W(z) = X(z)$$

$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - ... - a_N z^{-N} W(z) \quad (6.8.4)$$

Similarly by simplifying equation (6.8.3), we get

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + ... + b_M z^{-M} W(z) \quad (6.8.5)$$

Equation (6.8.4) and (6.8.5) can be realized together by a direct structure called direct form-II structure as shown in figure 6.8.2a. It uses less number of delay elements then the Direct Form I structure.

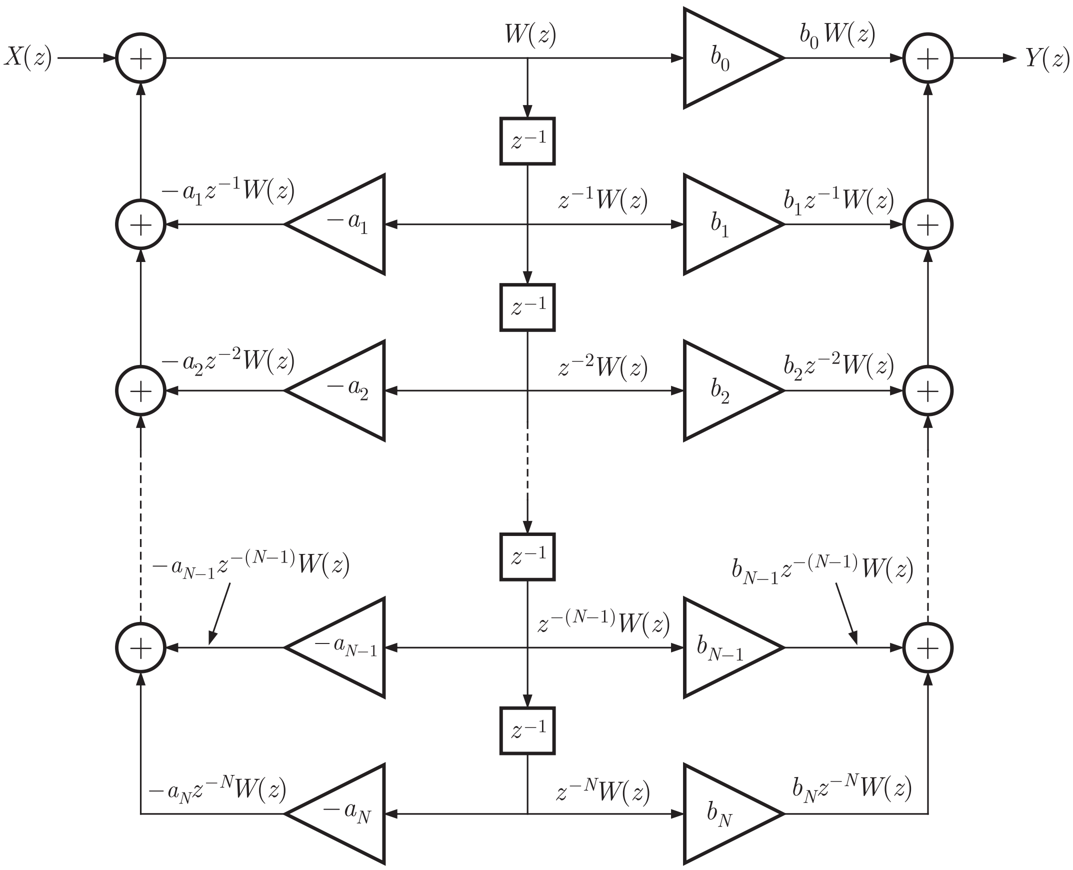


Fig 6.8.2a General structure of direct form-II realization

For example, consider the same transfer function $H(z)$ which is discussed above

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + 2z^{-2}}{1 + 4z^{-1} + 3z^{-2}}$$

Let $\dfrac{Y(z)}{X(z)} = \dfrac{Y(z)}{W(z)} \times \dfrac{W(z)}{X(z)}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 + 4z^{-1} + 3z^{-2}},$$

$$\frac{Y(z)}{W(z)} = 1 + 2z^{-1} + 2z^{-2}$$

so, $W(z) = X(z) - 4z^{-1}W(z) - 3z^{-2}W(z)$

and $Y(z) = 1W(z) + 2z^{-1}W(z) + 2z^{-2}W(z)$

Comparing these equations with standard form of equation (6.8.4) and (6.8.5), we have $a_1 = 4,\ a_2 = 3$ and $b_0 = 1, b_1 = 2, b_2 = 2$. Substitute these values in general structure of Direct form II , we get

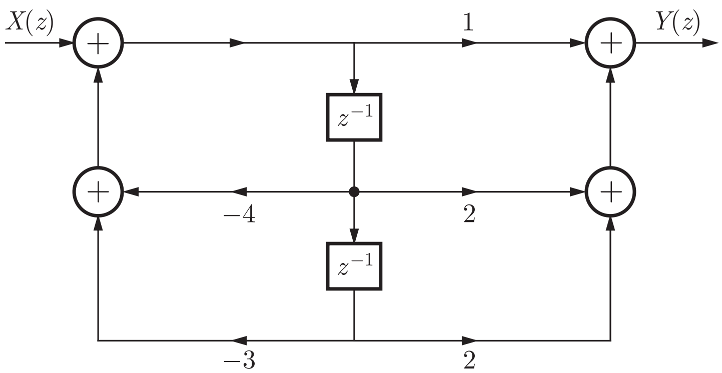


Fig 6.8.2b

6.8.3 Cascade Form

The transfer function $H(z)$ of a discrete time system can be expressed as a product of several transfer functions. Each of these transfer functions is realized in direct form-I or direct form II realization and then they are cascaded.

Consider a system with transfer function

$$H(z) = \frac{(b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2})(b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2})}{(1 + a_{k1}z^{-1} + a_{k2}z^{-2})(1 + a_{m1}z^{-1} + a_{m2}z^{-2})}$$

$$= H_1(z)\,H_2(z)$$

where $H_1(z) = \dfrac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$

$$H_2(z) = \frac{b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2}}{1 + a_{m1}z^{-1} + a_{m2}z^{-2}}$$

Realizing $H_1(z)$ and $H_2(z)$ in direct form II and cascading we obtain cascade form of the system function $H(z)$ as shown in figure 6.8.3.

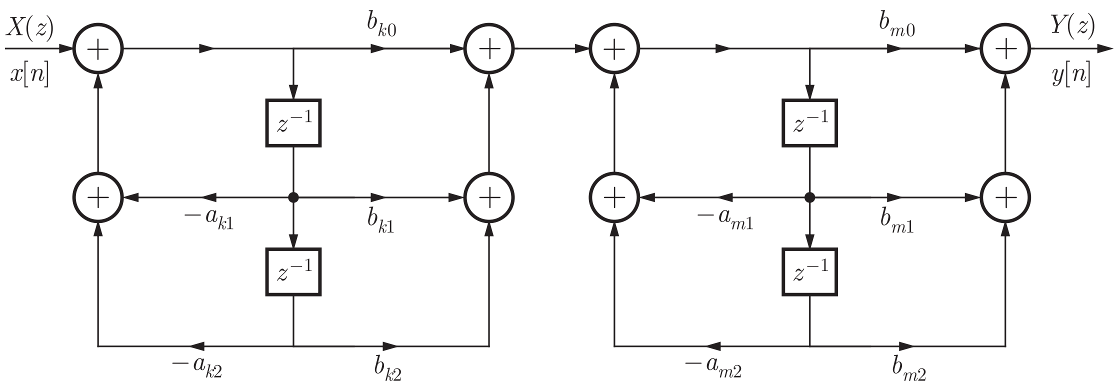


Fig 6.8.3 Cascaded form realization of discrete LTI system

6.8.4 Parallel Form

The transfer function $H(z)$ of a discrete time system can be expressed as the sum of several transfer functions using partial fractions. Then the individual transfer functions are realized in direct form I or direct form II realization and connected in parallel for the realization of $H(z)$. Let us consider the transfer function

$$H(z) = c + \frac{c_1}{1 - p_1 z^{-1}} + \frac{c_2}{1 - p_z z^{-1}} + ...\frac{c_N}{1 - p_n z^{-1}}$$

Now each factor in the system is realized in direct form II and connected in parallel as shown in figure 6.8.4.

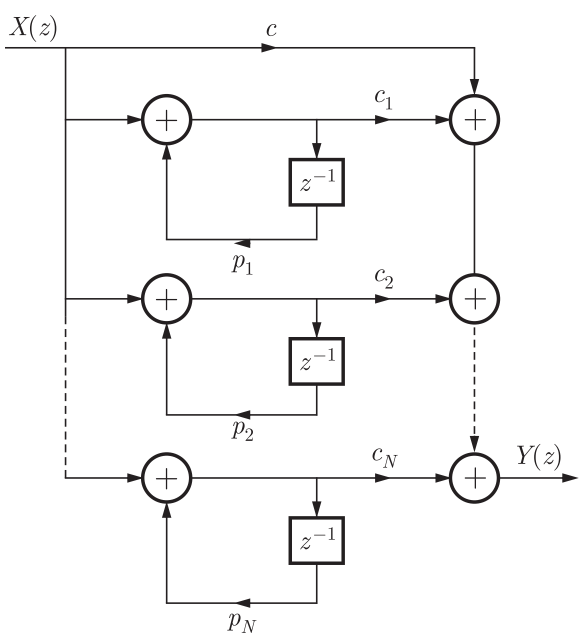


Fig 6.8.4 Parallel form realization of discrete LTI system

6.9 Relationship Between s-plane & z -plane

There exists a close relationship between the Laplace and z -transforms. We know that a DT sequence $x[n]$ is obtained by sampling a CT signal $x(t)$ with a sampling interval T , the CT sampled signal $x_s(t)$ is written as follows

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

where $x(nT)$ are sampled value of $x(t)$ which equals the DT sequence $x[n]$. By taking the Laplace transform of $x_s(t)$, we have

$$X(s) = L\{x_s(t)\} = \sum_{n=-\infty}^{\infty} x(nT)\,L\{\delta(t-nT)\}$$

$$= \sum_{n=-\infty}^{\infty} X(nT)\,e^{-nTs} \qquad (6.9.1)$$

The z-transform of $x[n]$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\,z^{-n} \qquad (6.9.2)$$

Comparing equation (6.9.1) and (6.9.2)

$$X(s) = X(z)\big|_{z=e^{sT}} \qquad \because\ x[n] = x(nT)$$

****

PRACTICE EXERCISE

LEVEL-1

MCQ 6.1 Consider a DT signal which is defined as follows

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

The z-transform of $x[n]$ will be

(A) $\frac{2z^{-1}}{z-1}$

(B) $\frac{2z}{2z-1}$

(C) $\frac{1}{z-\frac{1}{2}}$

(D) $\frac{1}{2-z}$

MCQ 6.2 If the z-transform of a sequence $x[n] = \{1, 1, -1, -\underset{\uparrow}{1}\}$ is $X(z)$, then what is the value of $X(1/2)$?

(A) 9

(B) -1.125

(C) 1.875

(D) 15

MCQ 6.3 The z-transform and its ROC of a discrete time sequence

$$x[n] = \begin{cases} -\left(\frac{1}{2}\right)^n, & n < 0 \\ 0, & n \geq 0 \end{cases}$$

will be

(A) $\frac{2z}{2z-1}, \ |z| > \frac{1}{2}$

(B) $\frac{z}{z-2}, \ |z| < \frac{1}{2}$

(C) $\frac{2z}{2z-1}, \ |z| < \frac{1}{2}$

(D) $\frac{2z^{-1}}{z-1}, \ |z| > \frac{1}{2}$

MCQ 6.4 The region of convergence of z-transform of the discrete time sequence

$$x[n] = \left(\frac{1}{2}\right)^{|n|} \text{ is}$$

(A) $\frac{1}{2} < |z| < 2$

(B) $|z| > 2$

(C) $-2 < |z| < 2$

(D) $|z| < \frac{1}{2}$

MCQ 6.5 Consider a discrete-time signal

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[-n-1]$$

The ROC of its z-transform is

(A) $3 < |z| < 2$

(B) $|z| < \frac{1}{2}$

(C) $|z| > \frac{1}{3}$

(D) $\frac{1}{3} < |z| < \frac{1}{2}$

MCQ 6.6 For a signal $x[n] = [\alpha^n + \alpha^{-n}]\, u[n]$, the ROC of its z-transform would be

(A) $|z| > \min\left(|\alpha|, \frac{1}{|\alpha|}\right)$

(B) $|z| > |\alpha|$

(C) $|z| > \max\left(|\alpha|, \frac{1}{|\alpha|}\right)$

(D) $|z| < |\alpha|$

MCQ 6.7 Match List I (discrete time sequence) with List II (z-transform) and choose the correct answer using the codes given below the lists

	List-I (Discrete time sequence)		**List-II (z-transform)**
P.	$u[n-2]$	**1.**	$\frac{1}{z^{-2}(1-z^{-1})}$, $|z| < 1$
Q.	$-u[-n-3]$	**2.**	$\frac{-z^{-1}}{1-z^{-1}}$, $|z| < 1$
R.	$u[n+4]$	**3.**	$\frac{1}{z^{-4}(1-z^{-1})}$, $|z| > 1$
S.	$u[-n]$	**4.**	$\frac{z^{-2}}{1-z^{-1}}$, $|z| > 1$

Codes :

	P	Q	R	S
(A)	1	4	2	3
(B)	2	4	1	3
(C)	4	1	3	2
(D)	4	2	3	1

MCQ 6.8 The z-transform of signal $x[n] = e^{jn\pi} u[n]$ is

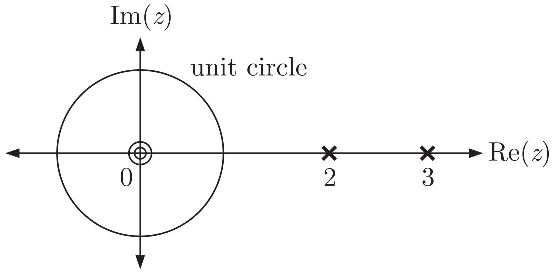
(A) $\frac{z}{z+1}$, ROC: $|z| > 1$

(B) $\frac{z}{z-j}$, ROC: $|z| > 1$

(C) $\frac{z}{z^2+1}$, ROC: $|z| < 1$

(D) $\frac{1}{z+1}$, ROC: $|z| < 1$

MCQ 6.9 Consider the pole zero diagram of an LTI system shown in the figure which corresponds to transfer function $H(z)$.



Match List I (The impulse response) with List II (ROC which corresponds to above diagram) and choose the correct answer using the codes given below:
$\{\text{Given that } H(1) = 1\}$

	List-I (Impulse response)		**List-II (ROC)**
P.	$[(-4)2^n + 6(3)^n]u[n]$	**1.**	does not exist
Q.	$(-4)2^n u[n] + (-6)3^n u[-n-1]$	**2.**	$\lvert z \rvert > 3$
R.	$(4)2^n u[-n-1] + (-6)3^n u[-n-1]$	**3.**	$\lvert z \rvert < 2$
S.	$4(2)^n u[-n-1] + (-6)3^n u[n]$	**4.**	$2 < \lvert z \rvert < 3$

Codes :

	P	Q	R	S
(A)	4	1	3	2
(B)	2	1	3	4
(C)	1	4	2	3
(D)	2	4	3	1

MCQ 6.10 The z-transform of a discrete time signal $x[n]$ is

$$X(z) = \frac{z+1}{z(z-1)}$$

What are the values of $x[0]$, $x[1]$ and $x[2]$ respectively ?

(A) 1, 2, 3 (B) 0, 1, 2

(C) 1, 1, 2 (D) -1, 0, 2

MCQ 6.11 The z-transform of a signal $x[n]$ is

$$X(z) = e^z + e^{1/z}, \; \lvert z \rvert \neq 0$$

$x[n]$ would be

(A) $\delta[n] + \frac{1}{\lfloor n}$ (B) $u[n] + \frac{1}{\lfloor n}$

(C) $u[n-1] + \lfloor n$ (D) $\delta[n] + \lfloor n-1$

Statement For Q. 12 - 14

Consider a discrete time signal $x[n]$ and its z-transform $X(z)$ given as

$$X(z) = \frac{z^2 + 5z}{z^2 - 2z - 3}$$

MCQ 6.12 If ROC of $X(z)$ is $|z| < 1$, then signal $x[n]$ would be

(A) $[-2(3)^n + (-1)^n]\,u[-n-1]$ (B) $[2(3)^n - (-1)^n]\,u[n]$

(C) $-2(3)^n u[-n-1] - (-1)^n u[n]$ (D) $[2(3)^n + 1]\,u[n]$

MCQ 6.13 If ROC of $X(z)$ is $|z| > 3$, then signal $x[n]$ would be

(A) $[2(3)^n - (-1)^n u[n]$ (B) $[-2(3)^n + (-1)^n]\,u[-n-1]$

(C) $-2(3)^n u[-n-1] - (-1)^n u[n]$ (D) $[2(3)^n + 1]\,u[n]$

MCQ 6.14 If ROC of $X(z)$ is $1 < |z| < 3$, the signal $x[n]$ would be

(A) $[2(3)^n - (-1)^n]\,u[n]$ (B) $[-2(3)^n + (-1)^n]\,u[-n-1]$

(C) $-2(3)^n u[-n-1] - (-1)^n u[n]$ (D) $[2(3)^n + (-1)^n]\,u[-n-1]$

MCQ 6.15 Consider a DT sequence

$$x[n] = x_1[n] + x_2[n]$$

where, $x_1[n] = (0.7)^n u[n-1]$ and

$x_2[n] = (-0.4)^n u[n-2]$

The region of convergence of z-transform of $x[n]$ is

(A) $0.4 < |z| < 0.7$ (B) $|z| > 0.7$

(C) $|z| < 0.4$ (D) none of these

MCQ 6.16 The z-transform of a DT signal $x[n]$ is

$$X(z) = \frac{z}{8z^2 - 2z - 1}$$

What will be the z-transform of $x[n-4]$?

(A) $\frac{(z+4)}{8(z+4)^2 - 2(z+4) - 1}$ (B) $\frac{z^5}{8z^2 - 2z - 1}$

(C) $\frac{4z}{128z^2 - 8z - 1}$ (D) $\frac{1}{8z^5 - 2z^4 - z^3}$

MCQ 6.17 If $x[n] = \alpha^n u[n]$, then the z-transform of $x[n+3]\,u[n]$ will be

(A) $\frac{z^{-2}}{z-\alpha}$ (B) $\frac{z^4}{z-\alpha}$

(C) $\alpha^3\left(\frac{z}{z-\alpha}\right)$ (D) $\frac{z^{-3}}{z-\alpha}$

MCQ 6.18 Let $x_1[n]$, $x_2[n]$ and $x_3[n]$ be three discrete time signals and $X_1(z)$, $X_2(z)$ and $X_3(z)$ are their z-transform respectively given as

$$X_1(z) = \frac{z^2}{(z-1)(z-0.5)},$$

$$X_2(z) = \frac{z}{(z-1)(z-0.5)}$$

and $$X_3(z) = \frac{1}{(z-1)(z-0.5)}$$

Then $x_1[n]$, $x_2[n]$ and $x_3[n]$ are related as

(A) $x_1[n-2] = x_2[n-1] = x_3[n]$ (B) $x_1[n+2] = x_2[n+1] = x_3[n]$

(C) $x_1[n] = x_2[n-1] = x_3[n-2]$ (D) $x_1[n+1] = x_2[n-1] = x_3[n]$

MCQ 6.19 The inverse z-transform of a function $X(z) = \dfrac{z^{-9}}{z-\alpha}$ is

(A) $\alpha^{n-10}u[n-10]$ (B) $\alpha^n u[n-10]$

(C) $\alpha^{n/10}u[n]$ (D) $\alpha^{n-9}u[n-9]$

MCQ 6.20 Let $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ be a z-transform pair, where

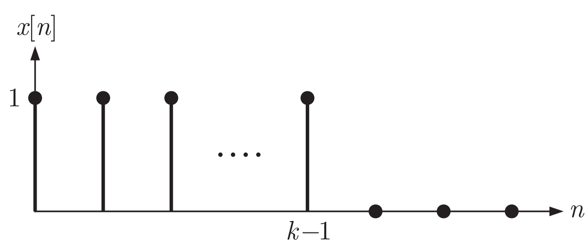
$$X(z) = \frac{z^{-2}}{z-3}$$

the value of $x[5]$ is

(A) 3 (B) 9

(C) 1 (D) 0

MCQ 6.21 The z-transform of the discrete time signal $x[n]$ shown in the figure is



(A) $\dfrac{z^{-k}}{1-z^{-1}}$ (B) $\dfrac{z^{-k}}{1+z^{-1}}$

(C) $\dfrac{1-z^{-k}}{1-z^{-1}}$ (D) $\dfrac{1+z^{-k}}{1-z^{-1}}$

MCQ 6.22 Consider the unilateral z-transform pair $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \dfrac{z}{z-1}$. The z-transform of $x[n-1]$ and $x[n+1]$ are respectively

(A) $\frac{z^2}{z-1}, \frac{1}{z-1}$ (B) $\frac{1}{z-1}, \frac{z^2}{z-1}$

(C) $\frac{1}{z-1}, \frac{z}{z-1}$ (D) $\frac{z}{z-1}, \frac{z^2}{z-1}$

MCQ 6.23 A discrete time causal signal $x[n]$ has the z-transform

$$X(z) = \frac{z}{z-0.4}, \quad \text{ROC}: |z| > 0.4$$

The ROC for z-transform of the even part of $x[n]$ will be

(A) same as ROC of $X(z)$ (B) $0.4 < |z| < 2.5$

(C) $|z| > 0.2$ (D) $|z| > 0.8$

MCQ 6.24 The z-transform of a discrete time sequence $y[n] = n[n+1]\,u[n]$ is

(A) $\frac{2z^2}{(z-1)^3}$ (B) $\frac{z(z+1)}{(z-1)^3}$

(C) $\frac{z}{(z-1)^2}$ (D) $\frac{1}{(z-1)^2}$

MCQ 6.25 Match List I (Discrete time sequence) with List II (z-transform) and select the correct answer using the codes given below the lists.

	List-I (Discrete time sequence)		**List-II (z-transform)**
P.	$n(-1)^n u[n]$	**1.**	$\frac{z^{-1}}{(1-z^{-1})^2}$, ROC: $|z| > 1$
Q.	$-nu[-n-1]$	**2.**	$\frac{1}{(1+z^{-1})}$, ROC: $|z| > 1$
R.	$(-1)^n u[n]$	**3.**	$\frac{z^{-1}}{(1-z^{-1})^2}$, ROC: $|z| < 1$
S.	$nu[n]$	**4.**	$-\frac{z^{-1}}{(1+z^{-1})^2}$, ROC: $|z| > 1$

Codes :

	P	Q	R	S
(A)	4	1	2	3
(B)	4	3	2	1
(C)	3	1	4	2
(D)	2	4	1	3

MCQ 6.26 A signal $x[n]$ has the following z-transform

$$X(z) = \log(1-2z), \quad \text{ROC}: |z| < 1/2$$

signal $x[n]$ is

(A) $\left(\frac{1}{2}\right)^n u[n]$ (B) $\frac{1}{n}\left(\frac{1}{2}\right)^n u[n]$

(C) $\frac{1}{n}\left(\frac{1}{2}\right)^n u[-n-1]$ (D) $\left(\frac{1}{2}\right)^n u[-n-1]$

MCQ 6.27 A discrete time sequence is defined as

$$x[n] = \frac{1}{n}(-2)^{-n} u[-n-1]$$

The z-transform of $x[n]$ is

(A) $\log\left(z+\frac{1}{2}\right), \quad \text{ROC}: |z| < \frac{1}{2}$ (B) $\log\left(z-\frac{1}{2}\right), \quad \text{ROC}: |z| < \frac{1}{2}$

(C) $\log(z-2), \quad \text{ROC}: |z| > 2$ (D) $\log(z+2), \quad \text{ROC}: |z| < 2$

MCQ 6.28 Consider a z-transform pair $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ with ROC R_x. The z transform and its ROC for $y[n] = a^n x[n]$ will be

(A) $X\left(\frac{z}{a}\right), \quad \text{ROC}: |a| R_x$ (B) $X(z+a), \quad \text{ROC}: R_x$

(C) $z^{-a} X(z), \quad \text{ROC}: R_x$ (D) $X(az), \quad \text{ROC}: |a| R_x$

MCQ 6.29 Let $X(z)$ be the z-transform of a causal signal $x[n] = a^n u[n]$ with ROC: $|z| > a$. Match the discrete sequences S_1, S_2, S_3 and S_4 with ROC of their z-transforms R_1, R_2 and R_3.

Sequences	**ROC**
S_1: $x[n-2]$	R_1: $\lvert z\rvert > a$
S_2: $x[n+2]$	R_2: $\lvert z\rvert < a$
S_3: $x[-n]$	R_3: $\lvert z\rvert < \frac{1}{a}$
S_4: $(-1)^n x[n]$	

(A) $(S_1, R_1), (S_2, R_2), (S_3, R_3), (S_4, R_3)$

(B) $(S_1, R_1), (S_2, R_1), (S_3, R_3), (S_4, R_1)$

(C) $(S_1, R_2), (S_2, R_1), (S_3, R_2), (S_4, R_3)$

(D) $(S_1, R_1), (S_2, R_2), (S_3, R_2), (S_4, R_3)$

MCQ 6.30 Consider a discrete time signal $x[n] = \alpha^n u[n]$ and its z-transform $X(z)$. Match List I (discrete signals) with List II (z-transform) and select the correct answer using the codes given below:

	List-I (Discrete time signal)		**List-II (z-transform)**
P.	$x[n/2]$	**1.**	$z^{-2}X(z)$
Q.	$x[n-2]\,u[n-2]$	**2.**	$X(z^2)$
R.	$x[n+2]\,u[n]$	**3.**	$X(z/\beta^2)$
S.	$\beta^{2n}x[n]$	**4.**	$\alpha^2X(z)$

Codes :

	P	Q	R	S
(A)	1	2	4	3
(B)	2	4	1	3
(C)	1	4	2	3
(D)	2	1	4	3

MCQ 6.31 Let $x[n]\xleftrightarrow{z}X(z)$ be a z-transform pair. Consider another signal $y[n]$ defined as

$$y[n]=\begin{cases}x[n/2], & \text{if } n \text{ is even}\\ 0, & \text{if } n \text{ is odd}\end{cases}$$

The z-transform of $y[n]$ is

(A) $\frac{1}{2}X(z)$ (B) $X(z^2)$

(C) $X(2z)$ (D) $X(z/2)$

MCQ 6.32 The z-transform of a discrete sequence $x[n]$ is $X(z)$, then the z-transform of $x[2n]$ will be

(A) $X(2z)$ (B) $X\left(\frac{z}{2}\right)$

(C) $\frac{1}{2}\left[X(\sqrt{z})+X(-\sqrt{z})\right]$ (D) $X(\sqrt{z})$

MCQ 6.33 Let $X(z)$ be z-transform of a discrete time sequence $x[n]=\left(-\frac{1}{2}\right)^n u[n]$

Consider another signal $y[n]$ and its z-transform $Y(z)$ given as

$$Y(z)=X(z^3)$$

What is the value of $y[n]$ at $n=4$?

(A) 0 (B) 2^{-12}

(C) 2^{12} (D) 1

MCQ 6.34 Consider a signal $x[n]$ and its z transform $X(z)$ given as

$$X(z)=\frac{4z}{8z^2-2z-1}$$

The z-transform of the following sequence will be

$$y[n]=x[0]+x[1]+x[2]+.....+x[n]$$

(A) $\dfrac{4z^2}{(z-1)(8z^2-2z-1)}$ (B) $\dfrac{4z(z-1)}{8z^2-2z-1}$

(C) $\dfrac{4z^2}{(z+1)(8z^2-2z-1)}$ (D) $\dfrac{4z(z+1)}{8z^2-2z-1}$

MCQ 6.35 Let $h[n]=\{1,2,0,-1,1\}$ and $x[n]=\{1,3,-1,-2\}$ be two discrete time sequences. What is the value of convolution $y[n]=h[n]*x[n]$ at $n=4$?

(A) -5 (B) 5

(C) -6 (D) -1

MCQ 6.36 What is the convolution of two DT sequence $x[n]=\{-\underset{\uparrow}{1},2,0,3\}$ and $h[n]=\{2,0,\underset{\uparrow}{3}\}$

(A) $\{-2,-4,\underset{\uparrow}{3},6,9\}$ (B) $\{-2,4,-\underset{\uparrow}{3},12,0,9\}$

(C) $\{9,6,\underset{\uparrow}{3},-4,-2\}$ (D) $\{-\underset{\uparrow}{3},6,7,4,6\}$

MCQ 6.37 If $x[n]\overset{\mathcal{Z}}{\longleftrightarrow}X(z)$ be a z-transform pair, then which of the following is true?

(A) $x^*[n]\overset{\mathcal{Z}}{\longleftrightarrow}X^*(-z)$ (B) $x^*[n]\overset{\mathcal{Z}}{\longleftrightarrow}-X^*(z)$

(C) $x^*[n]\overset{\mathcal{Z}}{\longleftrightarrow}X^*(z^*)$ (D) $x^*[n]\overset{\mathcal{Z}}{\longleftrightarrow}X^*(-z^*)$

MCQ 6.38 A discrete time sequence is defined as follows

$$x[n]=\begin{cases}1, & n \text{ is even}\\ 0, & \text{otherwise}\end{cases}$$

What is the final value of $x[n]$?

(A) 1 (B) 1/2

(C) 0 (D) does not exist

MCQ 6.39 Let $X(z)$ be the z-transform of a DT signal $x[n]$ given as

$$X(z)=\frac{0.5z^2}{(z-1)(z-0.5)}$$

The initial and final values of $x[n]$ are respectively

(A) 1, 0.5 (B) 0, 1

(C) 0.5, 1 (D) 1, 0

MCQ 6.40 A discrete-time system with input $x[n]$ and output $y[n]$ is governed by following difference equation

$$y[n]-\frac{1}{2}y[n-1]=x[n], \text{ with initial condition } y[-1]=3$$

The impulse response of the system

(A) $\frac{5}{2}\left(\frac{n}{2}-1\right),\ n\geq 0$ (B) $\frac{5}{2}\left(\frac{1}{2}\right)^n,\ n\geq 0$

(C) $\frac{5}{2}\left(\frac{1}{2}\right)^{n-1},\ n\geq 0$ (D) $\frac{5}{2}\left(\frac{1}{2}\right)^{n+1},\ n\geq 0$

MCQ 6.41 Consider a causal system with impulse response $h[n] = (2)^n u[n]$. If $x[n]$ is the input and $y[n]$ is the output to this system, then which of the following difference equation describes the system ?

(A) $y[n] + 2y[n+1] = x[n]$
(B) $y[n] - 2y[n-1] = x[n]$
(C) $y[n] + 2y[n-1] = x[n]$
(D) $y[n] - \frac{1}{2}y[n-1] = x[n]$

MCQ 6.42 The impulse response of a system is given as

$$h[n] = \delta[n] - \left(\frac{-1}{2}\right)^n u[n]$$

For an input $x[n]$ and output $y[n]$, the difference equation that describes the system is

(A) $y[n] + 2y[n-1] = 2x[n]$
(B) $y[n] + 0.5y[n-1] = 0.5x[n-1]$
(C) $y[n] + 2ny[n-1] = x[n]$
(D) $y[n] - 0.5y[n-1] = 0.5x[n-1]$

MCQ 6.43 The input-output relationship of a system is given as

$$y[n] - 0.4y[n-1] = x[n]$$

where, $x[n]$ and $y[n]$ are the input and output respectively. The zero state response of the system for an input $x[n] = (0.4)^n u[n]$ is

(A) $n(0.4)^n u[n]$
(B) $n^2(0.4)^n u[n]$
(C) $(n+1)(0.4)^n u[n]$
(D) $\frac{1}{n}(0.4)^n u[n]$

MCQ 6.44 A discrete time system has the following input-output relationship

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

If an input $x[n] = u[n]$ is applied to the system, then its zero state response will be

(A) $\left[\frac{1}{2} - (2)^n\right]u[n]$
(B) $\left[2 - \left(\frac{1}{2}\right)^n\right]u[n]$
(C) $\left[\frac{1}{2} - \left(\frac{1}{2}\right)^n\right]u[n]$
(D) $[2 - (2)^n]\,u[n]$

MCQ 6.45 Consider the transfer function of a system

$$H(z) = \frac{2z(z-1)}{z^2 + 4z + 4}$$

For an input $x[n] = 2\delta[n] + \delta[n+1]$, the system output is

(A) $2\delta[n+1] + 6(2)^n u[n]$
(B) $2\delta[n] - 6(-2)^n u[n]$
(C) $2\delta[n+1] - 6(-2)^n u[n]$
(D) $2\delta[n+1] + 6\left(\frac{1}{2}\right)^n u[n]$

MCQ 6.46 The signal $x[n] = (0.5)^n u[n]$ is when applied to a digital filter, it yields the following output

$$y[n] = \delta[n] - 2\delta[n-1]$$

If impulse response of the filter is $h[n]$, then what will be the value of sample $h[1]$?

(A) 1 (B) -2.5

(C) 0 (D) 0.5

MCQ 6.47 The transfer function of a discrete time LTI system is given as

$$H(z) = \frac{z}{z^2+1}, \quad \text{ROC}: |z| > 1$$

Consider the following statements

1. The system is causal and BIBO stable.
2. The system is causal but BIBO unstable.
3. The system is non-causal and BIBO unstable.
4. Impulse response $h[n] = \sin\left(\frac{\pi}{2}n\right)u[n]$

Which of the above statements are true ?

(A) 1 and 4 (B) 2 and 4

(C) 1 only (D) 3 and 4

MCQ 6.48 Which of the following statement is not true?

An LTI system with rational transfer function $H(z)$ is

(A) causal if the ROC is the exterior of a circle outside the outermost pole.

(B) stable if the ROC of $H(z)$ includes the unit circle $|z| = 1$.

(C) causal and stable if all the poles of $H(z)$ lie inside unit circle.

(D) none of above

MCQ 6.49 If $h[n]$ denotes the impulse response of a causal system, then which of the following system is not stable?

(A) $h[n] = n\left(\frac{1}{3}\right)^n u[n]$ (B) $h[n] = \frac{1}{3}\delta[n]$

(C) $h[n] = \delta[n] - \left(-\frac{1}{3}\right)^n u[n]$ (D) $h[n] = [(2)^n - (3)^n]\,u[n]$

MCQ 6.50 A causal system with input $x[n]$ and output $y[n]$ has the following relationship

$$y[n] + 3y[n-1] + 2y[n-2] = 2x[n] + 3x[n-1]$$

The system is

(A) stable (B) unstable

(C) marginally stable (D) none of these

MCQ 6.51 A causal LTI system is described by the following difference equation

$$y[n] = x[n] + y[n-1]$$

Consider the following statement

1. Impulse response of the system is $h[n] = u[n]$
2. The system is BIBO stable
3. For an input $x[n] = (0.5)^n u[n]$, system output is $y[n] = 2u[n] - (0.5)^n u[n]$

Which of the above statements is/are true?

(A) 1 and 2 (B) 1 and 3

(C) 2 and 3 (D) 1, 2 and 3

MCQ 6.52 Match List I (system transfer function) with List II (property of system) and choose the correct answer using the codes given below

List-I (System transfer function)		List-II (Property of system)	
P.	$H(z)=\frac{z^3}{(z-1.2)^3}$, ROC: $\lvert z\rvert>1.2$	**1.**	non causal but stable
Q..	$H(z)=\frac{z^2}{(z-1.2)^3}$, ROC: $\lvert z\rvert<1.2$	**2.**	neither causal nor stable
R.	$H(z)=\frac{z^4}{(z-0.8)^3}$, ROC: $\lvert z\rvert<0.8$	**3.**	causal but not stable
S.	$H(z)=\frac{z^3}{(z-0.8)^3}$, ROC: $\lvert z\rvert>0.8$	**4.**	both causal and stable

Codes :

	P	Q	R	S
(A)	4	2	1	3
(B)	1	4	2	3
(C)	3	1	2	4
(D)	3	2	1	4

MCQ 6.53 The transfer function of a DT feedback system is

$$H(z)=\frac{P}{1+P\left(\frac{z}{z-0.9}\right)}$$

The range of P, for which the system is stable will be

(A) $-1.9<P<-0.1$ (B) $P<0$

(C) $P>-1$ (D) $P>-0.1$ or $P<-1.9$

MCQ 6.54 Consider three stable LTI systems S_1, S_2 and S_3 whose transfer functions are given as

$$\boldsymbol{S_1}:\ H(z)=\frac{z-\frac{1}{2}}{z^2+\frac{1}{2}z-\frac{3}{16}}$$

$$\boldsymbol{S_2}:\ H(z)=\frac{z+1}{-\frac{2}{3}z^{-3}-\frac{1}{2}z^{-2}+\frac{4}{3}+z}$$

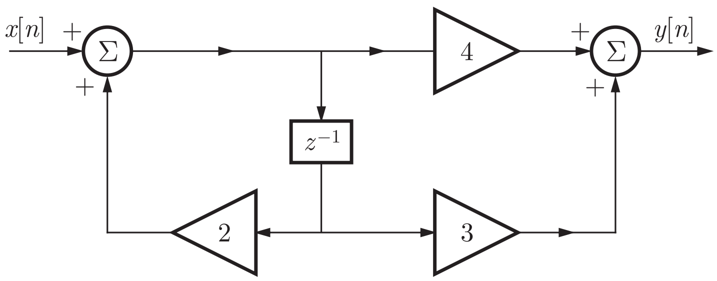
$$S_3 : H(z) = \frac{1 + \frac{1}{2}z^{-2} - \frac{4}{3}z^{-1}}{z^{-1}\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$

which of the above systems is/are causal?

(A) S_1 only (B) S_1 and S_2

(C) S_1 and S_3 (D) S_1, S_2 and S_3

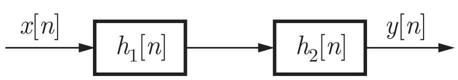
MCQ 6.55 The transfer function for the system realization shown in the figure will be



(A) $\frac{2z+3}{z-4}$ (B) $\frac{4z+3}{z-2}$

(C) $\frac{z+4}{2z-3}$ (D) $\frac{z+3}{z-2}$

MCQ 6.56 Consider a cascaded system shown in the figure



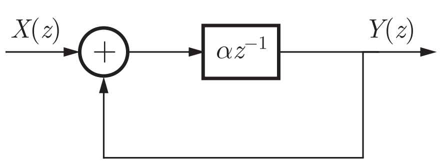
where, $h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1]$ and, $h_2[n] = \left(\frac{1}{2}\right)^n u[n]$

If an input $x[n] = \cos(n\pi)$ is applied, then output $y[n]$ equals to

(A) $\frac{1}{3}\cos(n\pi)$ (B) $\frac{5}{6}\cos(n\pi)$

(C) $\frac{13}{6}\cos(n\pi)$ (D) $\cos(n\pi)$

MCQ 6.57 The block diagram of a discrete time system is shown in the figure below



The range of α for which the system is BIBO stable, will be

(A) $\alpha > 1$ (B) $-1 < \alpha < 1$

(C) $\alpha > 0$ (D) $\alpha < 0$

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PRACTICE EXERCISE

LEVEL-2

MCQ 6.1 Let $x[n] = \delta[n-1] + \delta[n+2]$. The unilateral z - transform is

(A) z^{-2}
(B) z^2
(C) $-2z^{-2}$
(D) $2z^2$

MCQ 6.2 The unilateral z - transform of signal $x[n] = u[n+4]$ is

(A) $1 + z + z^2 + 3z + z^4$
(B) $\frac{1}{1-z}$
(C) $1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$
(D) $\frac{1}{1-z^{-1}}$

MCQ 6.3 The z transform of $\delta[n-k], k > 0$ is

(A) $z^k, z > 0$
(B) $z^{-k}, z > 0$
(C) $z^k, z \neq 0$
(D) $z^{-k}, z \neq 0$

MCQ 6.4 The z transform of $\delta[n+k], k > 0$ is

(A) $z^{-k}, z \neq 0$
(B) $z^k, z \neq 0$
(C) z^{-k}, all z
(D) z^k, all z

MCQ 6.5 The z transform of $u[n]$ is

(A) $\frac{1}{1-z^{-1}}, |z| > 1$
(B) $\frac{1}{1-z^{-1}}, |z| < 1$
(C) $\frac{z}{1-z^{-1}}, |z| < 1$
(D) $\frac{z}{1-z^{-1}}, |z| > 1$

MCQ 6.6 The z transform of $\left(\frac{1}{4}\right)^n (u[n] - u[n-5])$

(A) $\frac{z^5 - 0.25^5}{z^4(z-0.25)}, z > 0.25$
(B) $\frac{z^5 - 0.25^5}{z^4(z-0.25)}, z > 0$
(C) $\frac{z^5 - 0.25^5}{z^3(z-0.25)}, z < 0.25$
(D) $\frac{z^5 - 0.25^5}{z^4(z-0.25)}$, all z

MCQ 6.7 The z transform of is $\left(\frac{1}{4}\right)^4 u[-n]$ is

(A) $\frac{4z}{4z-1}, |z| > \frac{1}{4}$

(B) $\frac{4z}{4z-1}, |z| < \frac{1}{4}$

(C) $\frac{1}{1-4z}, |z| > \frac{1}{4}$

(D) $\frac{1}{1-4z}, |z| < \frac{1}{4}$

MCQ 6.8 The z transform of $3^n u[-n-1]$ is

(A) $\frac{z}{3-z}, |z| > 3$

(B) $\frac{z}{3-z}, |z| < 3$

(C) $\frac{3}{3-z}, |z| > 3$

(D) $\frac{3}{3-z}, |z| < 3$

MCQ 6.9 The z transform of $\left(\frac{2}{3}\right)^{|n|}$ is

(A) $\frac{-5z}{(2z-3)(3z-2)}, -\frac{3}{2} < z < -\frac{2}{3}$

(B) $\frac{-5z}{(2z-3)(3z-2)}, \frac{2}{3} < |z| < \frac{3}{2}$

(C) $\frac{5z}{(2z-3)(3z-2)}, \frac{2}{3} < |z| < \frac{2}{3}$

(D) $\frac{5z}{(2z-3)(3z-2)}, -\frac{3}{2} < z < -\frac{2}{3}$

MCQ 6.10 The z transform of $\cos\left(\frac{\pi}{3}n\right)u[n]$ is

(A) $\frac{z}{2}\frac{(2z-1)}{(z^2-z+1)},\ 0 < |z| < 1$

(B) $\frac{z}{2}\frac{(2z-1)}{(z^2-z+1)},\ |z| > 1$

(C) $\frac{z}{2}\frac{(1-2z)}{(z^2-z+1)},\ 0 < |z| < 1$

(D) $\frac{z}{2}\frac{(1-2z)}{(z^2-z+1)},\ |z| > 1$

MCQ 6.11 The z transform of $\{3,0,0,0,0,6,1,-4\}$

(A) $3z^5 + 6 + z^{-1} - 4z^{-2}, 0 \le |z| < \infty$

(B) $3z^5 + 6 + z^{-1} - 4z^{-2}, 0 < |z| < \infty$

(C) $3z^{-5} + 6 + z - 4z^2, 0 < |z| < \infty$

(D) $3z^{-5} + 6 + z - 4z^2, 0 \le |z| < \infty$

MCQ 6.12 The z transform of $x[n] = \{2,4,5,7,0,1\}$

(A) $2z^2 + 4z + 5 + 7z + z^3, z \neq \infty$

(B) $2z^{-2} + 4z^{-1} + 5 + 7z + z^3, z \neq \infty$

(C) $2z^{-2} + 4z^{-1} + 5 + 7z + z^3, 0 < |z| < \infty$

(D) $2z^2 + 4z + 5 + 7z^{-1} + z^{-3}, 0 < |z| < \infty$

MCQ 6.13 The z transform of $x[n] = \{1,0,-1,0,1,-1\}$ is

(A) $1 + 2z^{-2} - 4z^{-4} + 5z^{-5}$

(B) $1 - z^{-2} + z^{-4} - z^{-5}$

(C) $1 - 2z^2 + 4z^4 - 5z^5$

(D) $1 - z^2 + z^4 - z^5$

MCQ 6.14 The time signal corresponding to $\dfrac{z^2 - 3z}{z^2 + \frac{3}{2}z^{-1}}, \frac{1}{2} < |z| < 2$ is

(A) $-\dfrac{1}{2^n}u[n] - 2^{n+1}u[-n-1]$

(B) $-\dfrac{1}{2^n}u[n] - 2^{n+1}u[n+1]$

(C) $\dfrac{1}{2^n}u[n] + 2^{n+1}u[n+1]$

(D) $\dfrac{1}{2^n}u[n] - 2^{-n-1}u[-n-1]$

MCQ 6.15 The time signal corresponding to $\dfrac{3z^2 - \frac{1}{4}z}{z^2 - 16}, |z| > 4$ is

(A) $\left[\dfrac{49}{32}(-4)^n + \dfrac{47}{32}4^n\right]u[n]$

(B) $\left[\dfrac{49}{32}4^n + \dfrac{47}{32}4^n\right]u[n]$

(C) $\dfrac{49}{32}(-4)^n u[-n] + \dfrac{47}{32}4^n u[n]$

(D) $\dfrac{49}{32}4^n u[n] + \dfrac{47}{32}(-4)^n u[-n]$

MCQ 6.16 The time signal corresponding to $\dfrac{2z^4 - 2z^3 - 2z^2}{z^2 - 1}, |z| > 1$ is

(A) $2\delta[n-2] + [1 - (-1)^n]u[n-2]$

(B) $2\delta[n+2] + [1 - (-1)^n]u[n+2]$

(C) $2\delta[n+2] + [(-1)^n - 1]u[n+2]$

(D) $2\delta[n-2] + [(-1)^n - 1]u[n-2]$

MCQ 6.17 The time signal corresponding to $1 + 2z^{-6} + 4z^{-8}, |z| > 0$ is

(A) $\delta[n] + 2\delta[n-6] + 4\delta[n-8]$

(B) $\delta[n] + 2\delta[n+6] + 4\delta[n+8]$

(C) $\delta[-n] + 2\delta[-n+6] + 4\delta[-n+8]$

(D) $\delta[-n] + 2\delta[-n-6] + 4\delta[-n-8]$

MCQ 6.18 The time signal corresponding to $\sum_{k=5}^{10} \frac{1}{k} z^{-k}, |z| > 0$ is

(A) $\sum_{k=5}^{10} \frac{1}{k} \delta[n+k]$

(B) $\sum_{k=5}^{10} \frac{1}{k} \delta[n-k]$

(C) $\sum_{k=5}^{10} \frac{1}{k} \delta[-n+k]$

(D) $\sum_{k=5}^{10} \frac{1}{k} \delta[-n-k]$

MCQ 6.19 The time signal corresponding to $(1+z^{-1})^3$, $|z| > 0$ is

(A) $\delta[-n] + 3\delta[-n-1] + 3\delta[-n-2] + \delta[-n-3]$

(B) $\delta[-n] + 3\delta[-n+1] + 3\delta[-n+2] + \delta[-n+3]$

(C) $\delta[n] + 3\delta[n+1] + 3\delta[n+2] + \delta[n+3]$

(D) $\delta[n] + 3\delta[n-1] + 3\delta[n-2] + \delta[n-3]$

MCQ 6.20 The time signal corresponding to $z^6 + z^2 + 3 + 2z^{-3} + z^{-4}, |z| > 0$ is

(A) $\delta[n+6] + \delta[n+2] + 3\delta[n] + 2\delta[n-3] + \delta[n-4]$

(B) $\delta[n-6] + \delta[n-2] + 3\delta[n] + 2\delta[n+3] + \delta[n+4]$

(C) $\delta[-n+6] + \delta[-n+2] + 3\delta[-n] + 2\delta[-n+3] + \delta[-n+4]$

(D) $\delta[-n-6] + \delta[-n-2] + 3\delta[-n] + 2\delta[-n-3] + \delta[-n-4]$

MCQ 6.21 The time signal corresponding to $\frac{1}{1-\frac{1}{4}z^{-2}}, |z| > \frac{1}{4}$

(A) $\begin{cases} 2^{-n}, & n \text{ even and } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$

(B) $\left(\frac{1}{4}\right)^{2n} u[n]$

(C) $\begin{cases} 2^{-n}, & n \text{ odd}, n > 0 \\ 0, & n \text{ even} \end{cases}$

(D) $2^{-n} u[n]$

MCQ 6.22 The time signal corresponding to $\frac{1}{1-4z^{-2}}, |z| < \frac{1}{4}$ is

(A) $-\sum_{k=0}^{\infty} 2^{2(k+1)} \delta[-n-2(k+1)]$

(B) $-\sum_{k=0}^{\infty} 2^{2(k+1)} \delta[-n+2(k+1)]$

(C) $-\sum_{k=0}^{\infty} 2^{2(k+1)} \delta[n+2(k+1)]$

(D) $-\sum_{k=0}^{\infty} 2^{2(k+1)} \delta[n-2(k+1)]$

MCQ 6.23 The time signal corresponding to $\ln(1+z^{-1}), |z| > 0$ is

(A) $\frac{(-1)^{k-1}}{k} \delta[n-1]$

(B) $\frac{(-1)^{k-1}}{k} \delta[n+1]$

(C) $\frac{(-1)^{k}}{k} \delta[n-1]$

(D) $\frac{(-1)^{k}}{k} \delta[n+1]$

MCQ 6.24 If z - transform is given by

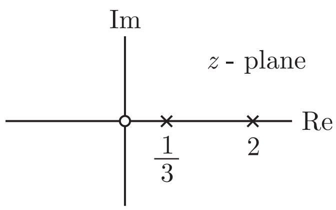
$$X(z) = \cos(z^{-3}), |z| > 0$$

The value of $x[12]$ is

(A) $-\frac{1}{24}$ (B) $\frac{1}{24}$

(C) $-\frac{1}{6}$ (D) $\frac{1}{6}$

MCQ 6.25 $X[z]$ of a system is specified by a pole zero pattern in below.



Consider three different solution of $x[n]$

$$x_1[n] \left[2^n - \left(\frac{1}{3}\right)^n\right] u[n]$$

$$x_2[n] = -2^n u[n-1] - \frac{1}{3^n} u[n]$$

$$x_3[n] = -2^n u[n-1] + \frac{1}{3^n} u[-n-1]$$

Correct solution is

(A) $x_1[n]$ (B) $x_2[n]$

(C) $x_3[n]$ (D) All three

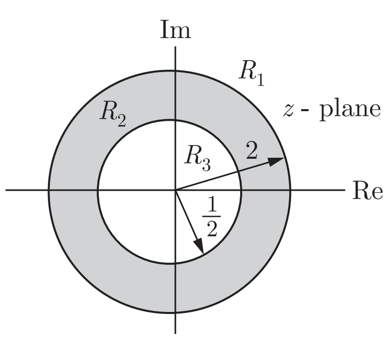
MCQ 6.26 Consider three different signal

$$x_1[n] = \left[2^n - \left(\frac{1}{2}\right)^n\right] u[n]$$

$$x_2[n] = -2^n u[-n-1] + \frac{1}{2^n} u[-n-1]$$

$$x_3[n] = -2^n u[-n-1] - \frac{1}{2^n} u[n]$$

Following figure shows the three different region. Choose the correct for the ROC of signal



	R_1	R_2	R_3
(A)	$x_1[n]$	$x_2[n]$	$x_3[n]$
(B)	$x_2[n]$	$x_3[n]$	$x_1[n]$
(C)	$x_1[n]$	$x_3[n]$	$x_2[n]$
(D)	$x_3[n]$	$x_2[n]$	$x_1[n]$

MCQ 6.27 Given the z transform

$$X(z) = \frac{1+\frac{7}{6}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$$

For three different ROC consider there different solution of signal $x[n]$:

(a) $|z| > \frac{1}{2}, x[n] = \left[\frac{1}{2^{n-1}} - \left(\frac{-1}{3}\right)^n\right]u[n]$

(b) $|z| < \frac{1}{3}, x[n] = \left[\frac{-1}{2^{n-1}} + \left(\frac{-1}{3}\right)^n\right]u[-n+1]$

(c) $\frac{1}{3} < |z| < \frac{1}{2}, x[n] = -\frac{1}{2^{n-1}}u[-n-1] - \left(\frac{-1}{3}\right)^n u[n]$

Correct solution are

(A) (a) and (b)

(B) (a) and (c)

(C) (b) and (c)

(D) (a), (b), (c)

MCQ 6.28 The $X(z)$ has poles at $z = \frac{1}{2}$ and $z = -1$. If $x[1] = 1x[-1] = 1$, and the ROC includes the point $z = \frac{3}{4}$. The time signal $x[n]$ is

(A) $\frac{1}{2^{n-1}}u[n] - (-1)^n u[-n-1]$

(B) $\frac{1}{2^n}u[n] - (-1)^n u[-n-1]$

(C) $\frac{1}{2^{n-1}}u[n] + u[-n+1]$

(D) $\frac{1}{2^n}u[n] + u[-n+1]$

MCQ 6.29 The $x[n]$ is right-sided, $X(z)$ has a signal pole, and $x[0] = 2$, $x[2] = \frac{1}{2}, x[n]$ is

(A) $\frac{u[-n]}{2^{n-1}}$

(B) $\frac{u[n]}{2^{n-1}}$

(C) $\frac{u[-n]}{2^{n+1}}$

(D) $a\frac{u[-n]}{2^{n+1}}$

MCQ 6.30 The z transform of $\left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[-n-1]$ is

(A) $\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}, \frac{1}{4} < |z| < \frac{1}{2}$

(B) $\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}}, \frac{1}{4} < |z| < \frac{1}{2}$

(C) $\dfrac{1}{1-\frac{1}{2}z^{-1}} - \dfrac{1}{1-\frac{1}{4}z^{-1}}, |z| > \dfrac{1}{2}$

(D) None of the above

Statement for Q. 31-36 :

Given the z - transform pair

$$x[n] \overset{z}{\longleftrightarrow} \frac{z^2}{z^2-16}, |z| < 4$$

MCQ 6.31 The z transform of the signal $x[n-2]$ is

(A) $\dfrac{z^4}{z^2-16}$

(B) $\dfrac{(z+2)^2}{(z+2)^2-16}$

(C) $\dfrac{1}{z^2-16}$

(D) $\dfrac{(z-2)^2}{(z-2)^2-16}$

MCQ 6.32 The z transform of the signal $y[n] = \dfrac{1}{2^n}x[n]$ is

(A) $\dfrac{(z+2)^2}{(x+2)^2-16}$

(B) $\dfrac{z^2}{z^2-4}$

(C) $\dfrac{(z-2)^2}{(z-2)^2-16}$

(D) $\dfrac{z^2}{z^2-64}$

MCQ 6.33 The z transform of the signal $x[-n] * x[n]$ is

(A) $\dfrac{z^2}{16z^2-257z^4-16}$

(B) $\dfrac{-16z^2}{(z^2-16)^2}$

(C) $\dfrac{z^2}{257z^2-16z^4-16}$

(D) $\dfrac{16z^2}{(z^2-16)^2}$

MCQ 6.34 The z transform of the signal $nx[n]$ is

(A) $\dfrac{32z^2}{(z^2-16)^2}$

(B) $\dfrac{-32z^2}{(z^2-16)^2}$

(C) $\dfrac{32z}{(z^2-16)^2}$

(D) $\dfrac{-32z}{(z^2-16)^2}$

MCQ 6.35 The z transform of the signal $x[n+1] + x[n-1]$ is

(A) $\dfrac{(z+1)^2}{(z+1)^2-16} + \dfrac{(z-1)^2}{(z-1)^2-16}$

(B) $\dfrac{z(z^2+1)}{z^2-16}$

(C) $\dfrac{z^2(-1+z)}{z^2-16}$

(D) None of the above

MCQ 6.36 The z transform of the signal $x[n] * x[n-3]$ is

(A) $\frac{z^{-3}}{(z^2-16)^2}$ (B) $\frac{z^7}{(z^2-16)^2}$

(C) $\frac{z^5}{(z^2-16)^2}$ (D) $\frac{z}{(z^2-16)^2}$

Statement for Q. 37-41 :

Given the z transform pair

$$3^n n^2 u[n] \xleftrightarrow{z} X(z)$$

MCQ 6.37 The time signal corresponding to $X(2z)$ is

(A) $n^2 3^n u[2n]$ (B) $\left(-\frac{3}{2}\right)^n n^2 u[n]$

(C) $\left(\frac{3}{2}\right)^n n^2 u[n]$ (D) $6^n n^2 u[n]$

MCQ 6.38 The time signal corresponding to $X(z^{-1})$ is

(A) $n^2 3^{-n} u[-n]$ (B) $n^2 3^{-n} u[-n]$

(C) $\frac{1}{n^2} 3^{\frac{1}{n}} u[n]$ (D) $\frac{1}{n^2} 3^{\frac{1}{n}} u[-n]$

MCQ 6.39 The time signal corresponding to $\frac{d}{dz}X(z)$ is

(A) $(n-1)^3 3^{n-1} u[n-1]$ (B) $n^3 3^n u[n-1]$

(C) $(1-n)^3 3^{n-1} u[n-1]$ (D) $(n-1)^3 3^{n-1} u[n]$

MCQ 6.40 The time signal corresponding to $\frac{z^2-z^{-2}}{2}X(z)$ is

(A) $\frac{1}{2}(x[n+2]-x[n-2])$ (B) $x[n+2]-x[n-2]$

(C) $\frac{1}{2}x[n-2]-x[n+2])$ (D) $x[n-2]-x[n+2]$

MCQ 6.41 The time signal corresponding to $\{X(z)\}^2$ is

(A) $[x[n]]^2$ (B) $x[n] * x[n]$

(C) $x(n) * x[-n]$ (D) $x[-n] * x[-n]$

MCQ 6.42 A causal system has input

$$x[n] = \delta[n] + \frac{1}{4}\delta[n-1] - \frac{1}{8}\delta[n-2] \text{ and output}$$

$$y[n] = \delta[n] - \frac{3}{4}\delta[n-1]$$

The impulse response of this system is

(A) $\frac{1}{3}\left[5\left(\frac{-1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n\right]u[n]$

(B) $\frac{1}{3}\left[5\left(\frac{1}{2}\right)^n + 2\left(\frac{-1}{4}\right)^n\right]u[n]$

(C) $\frac{1}{3}\left[5\left(\frac{1}{2}\right)^n - 2\left(\frac{-1}{4}\right)^n\right]u[n]$

(D) $\frac{1}{3}\left[5\left(\frac{1}{2}\right)^n + 2\left(\frac{1}{4}\right)^n\right]u[n]$

MCQ 6.43 A causal system has input $x[n] = (-3)^n u[n]$ and output

$$y[n] = \left[4(2)^n - \left(\frac{1}{2}\right)^n\right]u[n]$$

The impulse response of this system is

(A) $\left[7\left(\frac{1}{2}\right)^n - 10\left(\frac{1}{2}\right)^n\right]u[n]$

(B) $\left[7(2^n) - 10\left(\frac{1}{2}\right)^n\right]u[n]$

(C) $\left[10\left(\frac{1}{2}\right)^2 - 7(2)^n\right]u[n]$

(D) $\left[10(2^n) - 7\left(\frac{1}{2}\right)^n\right]u[n]$

MCQ 6.44 A system has impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$. The output $y[n]$ to the input $x[n]$ is given by $y[n] = 2\delta[n-4]$. The input $x[n]$ is

(A) $2\delta[-n-4] - \delta[-n-5]$

(B) $2\delta[n+4] - \delta[n+5]$

(C) $2\delta[-n+4] - \delta[-n+5]$

(D) $2\delta[n-4] - \delta[n-5]$

MCQ 6.45 A system is described by the difference equation

$$y[n] = x[n] - x[x-2] + x[n-4] - x[n-6]$$

The impulse response of system is

(A) $\delta[n] - 2\delta[n+2] + 4\delta[n+4] - 6\delta[n+6]$

(B) $\delta[n] + 2\delta[n-2] - 4\delta[n-4] + 6\delta[n-6]$

(C) $\delta[n] - \delta[n-2] + \delta[n-4] - \delta[n-6]$

(D) $\delta[n] - \delta[n+2] + \delta[n+4] - \delta[n+6]$

MCQ 6.46 The impulse response of a system is given by

$$h[n] = \frac{3}{4^n}u[n-1]$$

The difference equation representation for this system is

(A) $4y[n] - y[n-1] = 3x[n-1]$

(B) $4y[n] - y[n+1] = 3x[n+1]$

(C) $4y[n] + y[n-1] = -3x[n-1]$

(D) $4y[n] + y[n+1] = 3x[n+1]$

MCQ 6.47 The impulse response of a system is given by

$$h[n] = \delta[n] - \delta[n-5]$$

The difference equation representation for this system is

(A) $y[n] = x[n] - x[n-5]$

(B) $y[n] = x[n] - x[n+5]$

(C) $y[n] = x[n] + 5x[n-5]$

(D) $y[n] = x[n] - 5x[n+5]$

MCQ 6.48 Consider the following three systems

$$y_1[n] = 0.2y[n-1] + x[n] - 0.3x[n-1] + 0.02x[n-2]$$
$$y_2[n] = x[n] - 0.1x[n-1]$$
$$y_3[n] = 0.5y[n-1] + 0.4x[n] - 0.3x[n-1]$$

The equivalent system are

(A) $y_1[n]$ and $y_2[n]$ (B) $y_2[n]$ and $y_3[n]$

(C) $y_3[n]$ and $y_1[n]$ (D) all

MCQ 6.49 The z - transform function of a stable system is given as

$$H(z) = \frac{2 - \frac{3}{2}z^{-1}}{(1-2z^{-1})(1+\frac{1}{2}z^{-1})}$$

The impulse response $h[n]$ is

(A) $2^n u[-n+1] - \left(\frac{1}{2}\right)^n u[n]$ (B) $2^n u[-n-1] + \left(\frac{-1}{2}\right)^n u[n]$

(C) $-2^n u[-n-1] + \left(\frac{-1}{2}\right)^n u[n]$ (D) $2^n u[n] - \left(\frac{1}{2}\right)^n u[n]$

MCQ 6.50 The z-transform of a anti causal system is

$$X(z) = \frac{12-21z}{3-7z+12z^2}$$

The value of $x[0]$ is

(A) $-\frac{7}{4}$ (B) 0

(C) 4 (D) Does not exist

MCQ 6.51 The transfer function of a causal system is given as

$$H(z) = \frac{5z^2}{z^2-z-6}$$

The impulse response is

(A) $(3^n + (-1)^n 2^{n+1})u[n]$ (B) $(3^{n+1} + 2(-2)^n)u[n]$

(C) $(3^{n-1} + (-1)^n 2^{n+1})u[n]$ (D) $(3^{n-1} - (-2)^{n+1})u[n]$

MCQ 6.52 The transfer function of a system is given by

$$H(z) = \frac{z(3z-2)}{z^2-z-\frac{1}{4}}$$

The system is

(A) Causal and Stable

(B) Causal, Stable and minimum phase

(C) Minimum phase

(D) None of the above

MCQ 6.53 The z - transform of a signal $x[n]$ is given by

$$X(z) = \frac{3}{1 - \frac{10}{3}z^{-1} + z^{-2}}$$

If $X(z)$ converges on the unit circle, $x[n]$ is

(A) $-\frac{1}{3^{n-1}8}u[n] - \frac{3^{n+3}}{8}u[-n-1]$

(B) $\frac{1}{3^{n-1}8}u[n] - \frac{3^{n+3}}{8}u[-n]$

(C) $\frac{1}{3^{n-1}8}u[n] - \frac{3^{n+3}}{8}u[-n]$

(D) $-\frac{1}{3^{n-1}8}u[n] - \frac{3^{n+3}}{8}u[-n]$

MCQ 6.54 The transfer function of a system is given as

$$H(z) = \frac{4z^{-1}}{(1 - \frac{1}{4}z^{-1})^2}, |z| > \frac{1}{4}$$

The $h[n]$ is

(A) Stable

(B) Causal

(C) Stable and Causal

(D) None of the above

MCQ 6.55 The transfer function of a system is given as

$$H(z) = \frac{2(z + \frac{1}{2})}{(z - \frac{1}{2})(z - \frac{1}{3})}$$

Consider the two statements

Statement (i) : System is causal and stable.

Statement (ii) : Inverse system is causal and stable.

The correct option is

(A) (i) is true

(B) (ii) is true

(C) Both (i) and (ii) are true

(D) Both are false

MCQ 6.56 The system

$$y[n] = cy[n-1] - 0.12y[n-2] + x[n-1] + x[n-2]$$

is stable if

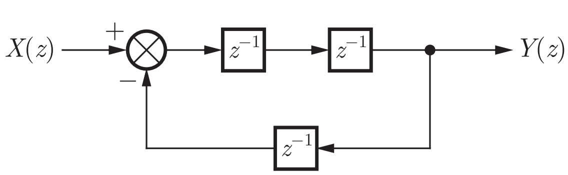
(A) $c < 1.12$

(B) $c > 1.12$

(C) $|c| < 1.12$

(D) $|c| > 1.12$

MCQ 6.57 The impulse response of the system shown below is



(A) $2^{(\frac{n}{2}-2)}(1 + (-1)^n)u[n] + \frac{1}{2}\delta[n]$

(B) $\frac{2^n}{2}(1 + (-1)^n)u[n] + \frac{1}{2}\delta[n]$

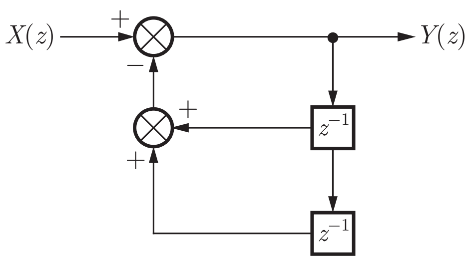
(C) $2^{(\frac{n}{2}-2)}(1+(-1)^n)\,u[n]-\frac{1}{2}\delta[n]$

(D) $\frac{2^n}{2}[1+(-1)^n]\,u[n]-\frac{1}{2}\delta[n]$

MCQ 6.58 The system diagram for the transfer function

$$H(z)=\frac{z}{z^2+z+1}$$

is shown below.



The system diagram is a

(A) Correct solution

(B) Not correct solution

(C) Correct and unique solution

(D) Correct but not unique solution

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